Chapter 14
The Quadratic Eigenvalue Problem

Thus far we have studied the damped system through its phase space matrix $A$ by taking into account its $J$-Hermitian structure and the underlying indefinite metric. Now we come back to the fact that this matrix stems from a second order system which carries additional structure commonly called ‘the quadratic eigenvalue problem’. We study the spectrum of $A$ and the behaviour of its exponential over time. A special class of so-called overdamped systems will be studied in some detail.

If we set $f = 0$ in (3.2), and make the substitution $y(t) = e^{\lambda t} y$, $y$ a constant vector, we obtain

$$Ay = \lambda y$$

(14.1)

and similarly, if in the homogeneous equation (1.1) we insert $x(t) = e^{\lambda t} x$, $x$ constant we obtain

$$(\lambda^2 M + \lambda C + K)x = 0$$

(14.2)

which is called the quadratic eigenvalue problem, attached to (1.1), $\lambda$ is an eigenvalue and $x$ a corresponding eigenvector. We start here a detailed study of these eigenvalues and eigenvectors constantly keeping the connection with the corresponding phase-space matrix.

Here, too, we can speak of the ‘eigenmode’ $x(t) = e^{\lambda t} x$ but the physical appeal is by no means as cogent as in the undamped case (Exercise 2.1) because the proportionality is a complex one. Also, the general solution of the homogeneous equation (1.1) will not always be given as a superposition of the eigenmodes.

Equations (14.1) and (14.2) are immediately seen to be equivalent via the substitution (for generality we keep on having complex $M, C, K$)

$$y = \begin{bmatrix} L_1^* x \\ \lambda L_2^* x \end{bmatrix}, \quad K = L_1 L_1^*, \quad M = L_2 L_2^*.$$  

(14.3)
Equation (14.2) may be written as
\[ Q(\lambda)x = 0, \]
where \( Q(\cdot) \), defined as
\[ Q(\lambda) = \lambda^2 M + \lambda C + K \quad (14.4) \]
is the quadratic matrix pencil associated with (1.1). The solutions of the equation \( Q(\lambda)x = 0 \) are referred to as the eigenvalues and the eigenvectors of the pencil \( Q(\cdot) \). For \( C = 0 \) the eigenvalue equation reduces to (2.3) with \( \lambda^2 = -\mu \). Thus, to two different eigenvalues \( \pm i\omega \) of (14.2) there corresponds only one linearly independent eigenvector. In this case the pencil can be considered as linear.

**Exercise 14.1** Show that in each of the cases
1. \( C = \alpha M \)
2. \( C = \beta K \)
the eigenvalues lie on a simple curve in the complex plane. Which are the curves?

The set \( X_\lambda = \{ x : Q(\lambda)x = 0 \} \) is the eigenspace for the eigenvalue \( \lambda \) of the pencil \( Q(\cdot) \), attached to (1.1). In fact, (14.3) establishes a one-to-one linear map from \( X_\lambda \) onto the eigenspace
\[ Y_\lambda = \{ y \in \mathbb{C}^{2n} : Ay = \lambda y \}, \]
in particular, \( \dim(Y_\lambda) = \dim(X_\lambda) \). In other words, ‘the eigenproblems (14.1) and (14.2) have the same geometric multiplicity’.

As we shall see soon, the situation is the same with the algebraic multiplicities. As is well known, the algebraic multiplicity of an eigenvalue of \( A \) is equal to its multiplicity as the root of the polynomial \( \det(\lambda I - A) \) or, equivalently, the dimension of the root space \( E_\lambda \). With \( J \) from (3.7) we compute
\[ JA - \lambda J = \begin{bmatrix} I & 0 \\ -L_2^{-1}L_1/\lambda & L_2^{-1} \end{bmatrix} \begin{bmatrix} -\lambda & 0 \\ 0 & \lambda^2 M + \lambda C + K \end{bmatrix} \begin{bmatrix} -L_1^*L_2^{-*}/\lambda \\ 0 \\ L_2^{-*} \end{bmatrix}, \quad (14.5) \]
where \( L_1, L_2 \) are from (14.3). Thus, after some sign manipulations,
\[ \det(\lambda I - A) = \det(L_2^{-1}(\lambda^2 M + \lambda C + K)L_2^{-*}) = \det Q(\lambda)/ \det M. \]
Hence the roots of the equation \( \det(\lambda I - A) = 0 \) coincide with those of
\[ \det(\lambda^2 M + \lambda C + K) = 0 \]