Chapter 17
Resonances and Resolvents

We have thus far omitted considering the inhomogeneous differential equation (1.1) since its solution can be obtained from the one of the homogeneous equation. There are, however, special types of right hand side $f$ in (1.1) the solution of which is particularly simply obtained and can, in turn, yield valuable information on the damped system itself. So they deserve a closer study.

Suppose that the external force $f$ is **harmonic**, that is,

$$f(t) = f_a \cos \omega t + f_b \sin \omega t,$$

where $\omega$ is a real constant and $f_a, f_b$ are real constant vectors. So called steady-state vibrations in which the system is continuously excited by forces whose amount does not vary much in time are oft approximated by harmonic forces.

With the substitution

$$x(t) = x_a \cos \omega t + x_b \sin \omega t$$

Equation (1.1) gives the linear system (here we keep $M, C, K$ real)

$$\begin{bmatrix} -\omega^2 M + K & \omega C \\ -\omega C & -\omega^2 M + K \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} = \begin{bmatrix} f_a \\ f_b \end{bmatrix}.$$  

The function (17.2) is called the **harmonic response** to the harmonic force (17.1). By introducing complex quantities

$$x_0 = x_a - ix_b, \quad f_0 = f_a - if_b$$

the system (17.3) is immediately seen to be equivalent to

$$Q(i\omega)x_0 = f_0.$$
By Proposition 15.5 we have the alternative

- The system is asymptotically stable or, equivalently, the system (17.4) is uniquely solvable.
- The system (17.4) becomes singular for some $\omega \in \sigma(\Omega)$, $\Omega$ from (2.5).

**Exercise 17.1** Show that $Q(i\omega)$ is non-singular whenever $\omega \not\in \sigma(\Omega)$.

**Exercise 17.2** Show that (17.4) is equivalent to

$$ (i\omega I - A)y_0 = F_0 \quad (17.5) $$

with

$$ y_0 = \begin{bmatrix} L_1^T x_0 \\ i\omega L_2^T x_0 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 0 \\ L_2^{-1} f_0 \end{bmatrix} $$

and $y(t) = y_0 e^{i\omega t}$ is a solution of $\dot{y} = Ay + F_0 e^{i\omega t}$.

**Exercise 17.3** Which harmonic response corresponds to a general right hand side vector $F_0$ in (17.5)?

From (17.5) we see that the harmonic response in the phase-space is given by the resolvent of $A$ taken on the imaginary axis. In analogy we call the function $Q(\lambda)^{-1}$ the *resolvent* of the quadratic pencil (14.4) or simply the *quadratic resolvent*. In the damping-free case $Q(i\omega)$ will be singular at every undamped frequency $\omega = \omega_j$.

**Proposition 17.4** If the system is not asymptotically stable then there is an $f(t)$ such that (1.1) has an unbounded solution.

**Proof.** There is a purely imaginary eigenvalue $i\omega$ and a real vector $x_0 \neq 0$ such that $Kx_0 = \omega^2 Mx_0$ and $Cx_0 = 0$. Take $f(t) = \alpha x_0 e^{i\omega t}$ and look for a solution $x(t) = \xi(t)x_0$. Substituted in (1.1) this gives

$$ \ddot{\xi} + \omega^2 \xi = \alpha e^{i\omega t} $$

which is known to have unbounded solutions. Q.E.D.

**Exercise 17.5** Prove the identity

$$ (\lambda I - A)^{-1} = \begin{bmatrix} \frac{1}{\lambda} I - \frac{L_1^T Q(\lambda)^{-1} L_1}{-L_2^T Q(\lambda)^{-1} L_2} & \frac{L_1^T Q(\lambda)^{-1} L_2}{L_2^T Q(\lambda)^{-1} L_2} \\ \frac{-L_2^T Q(\lambda)^{-1} L_1}{L_2^T Q(\lambda)^{-1} L_2} & \frac{L_2^T Q(\lambda)^{-1} L_1}{L_2^T Q(\lambda)^{-1} L_2} \end{bmatrix} \quad (17.6) $$

**Exercise 17.6** Show that the singularity at $\lambda = 0$ in (17.6) is removable by conveniently transforming the $1,1$- block.

Any system in which the frequency of the harmonic force corresponds to a purely imaginary eigenvalue as in Proposition 17.4 is said to be in the state of *resonance*. The same term is used, if an eigenvalue of (14.4) is close to