Chapter 4
The Singular Mass Case

If the mass matrix $M$ is singular no standard transformation to a first order system is possible. In fact, in this case we cannot prescribe some initial velocities and the phase-space will have dimension less than $2n$. This will be even more so, if the damping matrix, too, is singular; then we could not prescribe even some initial positions. In order to treat such systems we must first separate away these ‘inactive’ degrees of freedom and then arrive at phase-space matrices which have smaller dimension but their structure will be essentially the same as in the regular case studied before. Now, out of $M, C, K$ only $K$ is supposed to be positive definite while $M, C$ are positive semidefinite.

To perform the mentioned separation it is convenient to simultaneously diagonalise the matrices $M$ and $C$ which now are allowed to be only positive semidefinite.

Lemma 4.1 If $M, C$ are any real, symmetric, positive semidefinite matrices then there exists a real non-singular matrix $\Phi$ such that

$$\Phi^T M \Phi \quad \text{and} \quad \Phi^T C \Phi$$

are diagonal.

Proof. Suppose first that $\mathcal{N}(M) \cap \mathcal{N}(C) = \{0\}$. Then $M + C$ is positive definite and there is a $\Phi$ such that

$$\Phi^T (M + C) \Phi = I, \quad \Phi^T M \Phi = \mu,$$

$\mu$ diagonal. Then

$$\Phi^T C \Phi = I - \mu$$

is diagonal as well. Otherwise, let $u_1, \ldots, u_k$ be an orthonormal basis of $\mathcal{N}(M) \cap \mathcal{N}(C)$. Then there is an orthogonal matrix

$$U = \begin{bmatrix} \hat{U} & u_1 \cdots u_k \end{bmatrix}$$

such that

$$U^T M U = \begin{bmatrix} \hat{M} & 0 \\ 0 & 0 \end{bmatrix}, \quad U^T C U = \begin{bmatrix} \hat{C} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\mathcal{N}(\hat{M}) \cap \mathcal{N}(\hat{C}) = \{0\}$ and there is a non-singular $\hat{\Phi}$ such that $\hat{\Phi}^T \hat{M} \hat{\Phi}$ and $\hat{\Phi}^T \hat{C} \hat{\Phi}$ diagonal. Now set

$$\Phi = U \begin{bmatrix} \hat{\Phi} & 0 \\ 0 & 0 \end{bmatrix}.$$  

Q.E.D.

We now start separating the ‘inactive’ variables. Using the previous lemma there is a non-singular $\Phi$ such that

$$M' = \Phi^T M \Phi = \begin{bmatrix} M'_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C' = \Phi^T C \Phi = \begin{bmatrix} C'_1 & 0 & 0 \\ 0 & C'_2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $M'_1$, $C'_2$ are positive semidefinite ($C'_2$ may be lacking). Then set

$$K' = \Phi^T K \Phi = \begin{bmatrix} K'_{11} & K'_{12} & K'_{13} \\ K'_{12} & K'_{22} & K'_{23} \\ K'_{13} & K'_{23} & K'_{33} \end{bmatrix},$$

where $K'_{33}$ is positive definite (as a principal submatrix of the positive definite $K$). By setting

$$x = \Phi x', \quad x' = \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}, \quad K' = \Phi^T K \Phi$$

we obtain an equivalent system

\begin{align*}
M'_1 \ddot{x}'_1 + C'_1 \dot{x}'_1 + K'_{11} x'_1 + K'_{12} x'_2 + K'_{13} x'_3 &= \varphi_1 \\
C'_2 \ddot{x}'_2 + K'_{12} \dot{x}'_1 + K'_{22} x'_2 + K'_{23} x'_3 &= \varphi_2 \\
K'_{13} \dot{x}'_1 + K'_{23} x'_2 + K'_{33} x'_3 &= \varphi_3 
\end{align*}  \tag{4.1}