Perturbation Theory: Feshbach-Schur Method

As we have seen, many basic questions of quantum dynamics can be reduced to finding and characterizing the spectrum of the appropriate Schrödinger operator. Though this task, known as spectral analysis, is much simpler than the task of analyzing the dynamics directly, it is far from trivial. The problem can be greatly simplified if the Schrödinger operator $H$ under consideration is close to an operator $H_0$ whose spectrum we already know. In other words, the operator $H$ is of the form $H = H_0 + \kappa W$, where

$$H_\kappa = H_0 + \kappa W,$$  \hfill (11.1)

$H_0$ is an operator at least part of whose spectrum is well understood, $\kappa$ is a small real parameter called the coupling constant, and $W$ is an operator, called the perturbation. (All the operators here are assumed to be self-adjoint.)

If the operator $W$ is bounded relative to $H_0$, say in the sense that $D(H_0) \subset D(W)$ and

$$\|Wu\| \leq c\|H_0u\| + c'|u|$$

for some $c, c' \geq 0$, for all $u \in D(H_0)$, \hfill (11.2)

then “standard” perturbation theory applies, and allows us to find or estimate eigenvalues of $H_\kappa$ (see [RSIV, Ka, HS]).

In this chapter, we describe a powerful technique which allows us to estimate eigenvalues of $H_\kappa$ even in cases where $W$ is not bounded in terms of $H_0$. This is important in applications. We consider several examples of applications of this method, one of which is the hydrogen atom in a weak constant magnetic field $B$. Combining the expressions derived in Sections 7.5 and 7.7, we see that the Schrödinger operator for such an atom is

$$H_B := \frac{1}{2m}(p - eA)^2 - \frac{e^2}{|x|},$$  \hfill (11.3)

where $e < 0$ denotes the electron charge, and we have kept units in which the speed of light is $c = 1$. Recall that the vector potential $A$ is related to
the magnetic field $B$ by $B = \nabla \times A$. In the notation of Section 7.7, $H_B = H(A, -e^2/|x|)$. Expanding the square in (11.3), we find

$$H_B = H_0 + W_B$$

(11.4)

where $H_0 = \frac{1}{2m}p^2 - \frac{e^2}{|x|}$ is the Schrödinger operator of the hydrogen atom (see Section 7.5), and

$$W_B = \frac{|e|}{m} A \cdot p + \frac{e^2}{2m} |A|^2.$$

(11.5)

Here we have assumed the gauge condition $\nabla \cdot A \equiv 0$. The small parameter $\kappa$ here is the strength $|B|$ of the magnetic field.

Now we know from Section 7.5 that the operator $H_0$ has a series of eigenvalues

$$E_n := -\left(\frac{me^4}{2\hbar^2}\right) \frac{1}{n^2}, \quad n = 1, 2, \ldots$$

as well as continuous spectrum in $[0, \infty)$. One would expect that for weak magnetic fields $B$, the operator $H_B$ has eigenvalues $E_{B,n}$ close to $E_n$, at least for the few smallest $E_n$’s. This is not so obvious as it might seem at first glance, since the perturbation $W_B$ is not bounded relative to $H_0$—i.e., (11.2) does not hold for any $c$ and $c'$, small or large. (The perturbation (11.5) grows in $x$: take for example $A := \frac{B}{2}(-x_2, x_1, 0)$.) The method we present below does show rigorously that such eigenvalues exist, though we will make only formal computations of $E_{B,n}$.

Two other examples we present below display different physical phenomena, which conceptually and technically are considerably more complicated.

### 11.1 The Feshbach-Schur Method

Before returning to our perturbation problem, we state a general result used below, which allows us to reduce a perturbation problem on a large space to one on a small space. (For some motivating discussion see Appendix 11.5.) Let $P$ and $\bar{P}$ be orthogonal projections (i.e. $P, \bar{P}$ are self-adjoint, $P^2 = P$, and $\bar{P}^2 = \bar{P}$) on a separable Hilbert space $X$, satisfying $P + \bar{P} = 1$. Let $H$ be a self-adjoint operator on $X$. We assume that $\text{Ran}P \subset D(H)$, that $H_{\bar{P}} := PHP |_{\text{Ran} \bar{P}}$ is invertible, and

$$\|R_P\| < \infty, \quad \|PHR_{\bar{P}}\| < \infty \quad \text{and} \quad \|R_{\bar{P}}HP\| < \infty,$$

(11.6)

where $R_P = \bar{P}H_{\bar{P}}^{-1}P$. We define the operator

$$F_P(H) := P(H - HR_{\bar{P}}H)P |_{\text{Ran} P}.$$

(11.7)

The key result for us is the following: