In this chapter, we derive a convenient representation for the integral kernel of the Schrödinger evolution operator, $e^{-iHt/\hbar}$. This representation, the “Feynman path integral”, will provide us with a heuristic but effective tool for investigating the connection between quantum and classical mechanics. This investigation will be undertaken in the next section.

### 14.1 The Feynman Path Integral

Consider a particle in $\mathbb{R}^d$ described by a self-adjoint Schrödinger operator

$$H = -\frac{\hbar^2}{2m}\Delta + V(x).$$

Recall that the dynamics of such a particle is given by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi.$$

Recall also that the solution to this equation, with the initial condition $\psi|_{t=0} = \psi_0$, is given in terms of the evolution operator $U(t) := e^{-iHt/\hbar}$ as

$$\psi = U(t)\psi_0.$$ 

Our goal in this section is to understand the evolution operator $U(t) = e^{-iHt/\hbar}$ by finding a convenient representation of its integral kernel. We denote the integral kernel of $U(t)$ by $U_t(y, x)$ (also called the propagator from $x$ to $y$).

A representation of the exponential of a sum of operators is provided by the Trotter product formula (Theorem 14.2) which is explained in Section 14.3 at the end of this chapter. The Trotter product formula says that

$$e^{-iHt/\hbar} = e^{i(\frac{\hbar^2}{2m}\Delta - Vt)/\hbar} = \text{s-lim}_{n \to \infty} K_n^n$$
where \( K_n := e^{i \frac{\Delta}{2mn}} e^{-i \frac{V}{\hbar}} \). Let \( K_n(x, y) \) be the integral kernel of the operator \( K_n \). Then by Proposition 23.12,

\[
U_t(y, x) = \lim_{n \to \infty} \int \cdots \int K_n(y, x_{n-1}) \cdots K_n(x_2, x_1)K_n(x_1, x)dx_{n-1} \cdots dx_1.
\]

(14.1)

Now (see Section 23.3)

\[
K_n(y, x) = e^{i \frac{\Delta}{2mn}} (y, x)e^{-i \frac{V(y)t}{\hbar}}
\]

since \( V \), and hence \( e^{-i \frac{Vt}{\hbar}} \), is a multiplication operator (check this).

Using the expression (2.22), and plugging into (14.1) gives us

\[
U_t(y, x) = \lim_{n \to \infty} \int \cdots \int e^{iS_n/\hbar} \left( \frac{2\pi i \hbar}{mn} \right)^{-nd/2} dx_1 \cdots dx_{n-1}
\]

where

\[
S_n := \sum_{k=0}^{n-1} (mn|x_{k+1} - x_k|^2/2t - V(x_{k+1})t/n)
\]

with \( x_0 = x, x_n = y \). Define the piecewise linear function \( \phi_n \) such that \( \phi_n(0) = x, \phi_n(t/n) = x_1, \cdots, \phi_n(t) = y \) (see Fig. 14.1).

\[
\phi_n
\]

\[
y
\]

\[
x
\]

\[
\frac{t}{n} \quad \frac{2t}{n} \quad \cdots \quad \frac{(n-1)t}{n} \quad t
\]

Fig. 14.1. Piecewise linear function.

Then

\[
S_n = \sum_{k=0}^{n-1} \left\{ \frac{m|\phi_n((k+1)t/n) - \phi_n(kt/n)|^2}{2(t/n)^2} - V(\phi_n((k+1)t/n)) \right\} t/n.
\]

Note that \( S_n \) is a Riemann sum for the classical action

\[
S(\phi, t) = \int_0^t \left\{ \frac{m}{2} |\phi(s)|^2 - V(\phi(s)) \right\} ds
\]