Quaternions: A Mathematica Package for Quaternionic Analysis*

M.I. Falcão and Fernando Miranda

[1] Departamento de Matemática e Aplicações, Universidade do Minho
mif@math.uminho.pt
[2] Departamento de Matemática e Aplicações and Centro de Matemática, Universidade do Minho
fmiranda@math.uminho.pt

Abstract. This paper describes new issues of the Mathematica standard package `Quaternions` for implementing Hamilton's Quaternion Algebra. This work attempts to endow the original package with the ability to perform operations on symbolic expressions involving quaternion-valued functions. A collection of new functions is introduced in order to provide basic mathematical tools necessary for dealing with regular functions in $\mathbb{R}^{n+1}$, for $n \geq 2$. The performance of the package is illustrated by presenting several examples and applications.

Keywords: Quaternions, Clifford Analysis, monogenic functions, symbolic computation.

1 Introduction

Quaternions were introduced in 1843 by the Irish mathematician William Rowan Hamilton. One of the most popular application of Hamilton's Algebra is concerned with the use of quaternions for describing 3D rotations. In fact, *quaternions are inextricably linked to rotations* ([4]) and their use has become indispensable in all high technologies with need of calculations in real time.

Nowadays, with the development of Quaternionic Analysis, quaternions are also recognized as a powerful tool for modeling and solving problems in both theoretical and applied mathematics ([20]).

The increasing interest in using quaternions and their applications in almost all applied sciences has motivated the emergence of several software packages to perform computations in the algebra of the real quaternions (see, for example, [15,16,22]), or more generally, in Clifford Algebras (see [1,3] and the references therein for details).

Three main reasons lead us to develop this work:

- to endow the standard package `Quaternions` with the ability to perform operations on quaternion-valued functions;
- to extend the applicability of the package to arbitrary dimensions;
- to introduce a basic set of special polynomials, which plays an important role in applications.

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2 Algebra of Quaternions

2.1 Basic Results

Any quaternion x can be written in the form

$$x = x_0 + ix_1 + jx_2 + kx_3,\ x_i \in \mathbb{R}\,, \tag{1}$$

where Hamilton's imaginary units i, j and k satisfy the multiplication rules

$$i^2 = j^2 = k^2 = -1 \text{ and } ij = -ji = k\,. \tag{2}$$

This non-commutative product generates the algebra of real quaternions $\mathbb{H}$. The real vector space $\mathbb{R}^4$ will be embedded in $\mathbb{H}$ by identifying the element $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ with the element $x = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$. Thus, throughout this paper, we will use the same symbol x to represent a point in $\mathbb{R}^4$ and the corresponding quaternion in $\mathbb{H}$.

For a quaternion x of the form (1) we will distinguish between the real part of x,

$$\operatorname{Re} x := x_0\,,$$

and the vector part of x,

$$\operatorname{Vec} x = \underline{x} := ix_1 + jx_2 + kx_3\,,$$

so that a quaternion x can be written as

$$x = x_0 + \underline{x}\,.$$

When $x = \underline{x}$, x is called a *pure quaternion*. The conjugate of x is

$$\bar{x} := x_0 - \underline{x}$$

and the norm of x, $|x|$, is defined by

$$|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2\,.$$

If $|x| = 1$, x is said to be a *unit quaternion*. It immediately follows that each non-zero $x \in \mathbb{H}$ has an inverse given by

$$x^{-1} = \frac{\bar{x}}{|x|^2}$$

and therefore $\mathbb{H}$ is a non-commutative division ring or a skew field.

We note that an arbitrary non-null quaternion x can be written as

$$x = x_0 + \omega(x)|\underline{x}|\,, \tag{3}$$

where $\omega(x)$ is the unit quaternion

$$\omega(x) = \frac{\underline{x}}{|\underline{x}|}\,, \tag{4}$$