Chapter 6
Transfer Function Models and the Limitations of Recursive Least Squares

In the first part of this book, we have considered in some detail the recursive estimation of parameters in linear regression models. Such models are, however, primarily utilized in the evaluation of static relationships between variables and, while they may well figure in some aspects of systems analysis (for example the characterization of equilibrium or steady state behaviour) they are not generally suitable for use in dynamic systems analysis. Of course, dynamic systems can be modelled in various ways and so consideration of methods for estimating the parameters of dynamic system models is dependent, to some extent at least, on the nature of the model chosen to characterize the system. This chapter considers a specific class of models for linear stochastic dynamic systems and shows how the recursive methods of estimation discussed in the previous chapter can be modified to handle the estimation of the parameters that characterize these models.

In the classical statistical literature, the models considered here have been termed ‘time series’ representations since they attempt to describe a time series of data, i.e. data arranged in temporal order (Box and Jenkins, 1970). While our treatment of these models will be similar to that used in statistical texts, its emphasis will be rather different: the classical approach tends to concentrate on the stochastic aspects of the time series model; while here we emphasize the relationship between the assumed deterministic input or ‘exogenous’ variables and that part of the measured output of the stochastic dynamic system that can be ‘explained’ in relation to these inputs. This emphasis arises because we are concerned with the mathematical characterization of stochastic dynamic systems for purposes such as: the better understanding of causal dynamic relationships in physical systems; the use of models for forecasting the outcome of changes in the exogenous input variables; and for designing control and management schemes.

The reader has already been introduced to one type of linear stochastic dynamic model in the last two chapters: namely the discrete time Gauss-Markov or state-space model used both as the vehicle for the development of the TVP estimation algorithms and the KFX version of the the Kalman Filter algorithm in section 4.4 of chapter 4. This seems, therefore, an appropriate starting point for our present discussion. If we consider the state space model (4.46) used in relation to the KFX
algorithm, then this will serve to emphasize that we are concerned here with the modeling of a general, linear, stochastic dynamic system; rather than considering the model simply as a device for representing the variations in parameters. In this manner, we can retain the Gauss-Markov model (4.23) of chapter 4 for the latter purpose and consider its use as a model for time-variable parameter variations in later chapters.

6.1 Introduction: Direct Estimation of the State Space Model

To begin with, assume that the dynamic system is stable and stationary, so that the relevant state space description is (4.46): i.e.

\[
\begin{align*}
x(k) &= Ax(k-1) + Bu(k-1) + D\zeta(k-1) \\
y(k) &= Cx(k) + B_Iu(k) + \zeta(k)
\end{align*}
\]

(6.1)

where the matrices A, B, C, D and B_I are now composed of time-invariant real numbers (to ensure stationarity). It is further assumed that the system is stable, with the eigenvalues of the characteristic equation

\[
\det[I - Az^{-1}] = 1 + a_1z^{-1} + \ldots + a_nz^{-n} = 0
\]

(6.2)

all lying outside the unit circle in the complex z^{-1} plane (i.e. equivalent to the roots of the equation \(z^n + a_1z^{n-1} + \ldots + a_n = 0\) lying inside the unit circle in the complex z plane).

Nominally, the problem of parameter estimation for the model (6.1) is to infer, in some statistical manner, the parameters of the A, B, C, D and B_I matrices, on the basis of the measurements u(k) and y(k) over some observational interval, say \(k = 1, 2, \ldots, N\). In his 1960 paper, Kalman recognized this as a difficult and, at that time, unsolved problem when he noted that his approach to state variable estimation was limited by its assumption that the model parameters were known \textit{a priori}. He concluded:

... \{it is\} convenient to start with the model and regard the problem of obtaining the model itself as a separate question. To be sure, the two problems should be optimized jointly if possible; the author is not aware, however, of any study of the joint optimization problem.

This challenge was taken up quickly and several authors suggested approaches to recursive parameter estimation that were strongly stimulated by the nature of the KF algorithm. Perhaps the best known outcome of this research effort is the \textit{Extended Kalman Filter} (EKF), first suggested by Kopp and Orford (1963). This is a relatively straightforward approach, in which the state vector \(x(k)\) is augmented to include an unknown parameter vector which includes any unknown elements in the model matrices A, B, C, D and B_I; estimation then proceeds using a Kalman filter-like algorithm obtained by linearizing the now non-linear relationships (arising from the products between the unknown parameters and the unknown states)