Chapter 2
Basic Topological Properties
of Finite Spaces

In this chapter we present some results concerning elementary topological aspects of finite spaces. The proofs use basic elements of Algebraic Topology and have a strong combinatorial flavour. We study further homotopical properties including classical homotopy invariants and finite analogues of well-known topological constructions.

2.1 Homotopy and Contiguity

Recall that two simplicial maps \( \varphi, \psi : K \to L \) are said to be \textit{contiguous} if for every simplex \( \sigma \in K \), \( \varphi(\sigma) \cup \psi(\sigma) \) is a simplex of \( L \). Two simplicial maps \( \varphi, \psi : K \to L \) lie in the same \textit{contiguity class} if there exists a sequence \( \varphi = \varphi_0, \varphi_1, \ldots, \varphi_n = \psi \) such that \( \varphi_i \) and \( \varphi_{i+1} \) are contiguous for every \( 0 \leq i < n \).

If \( \varphi, \psi : K \to L \) lie in the same contiguity class, the induced maps in the geometric realizations \( |\varphi|, |\psi| : |K| \to |L| \) are homotopic (see Corollary A.1.3 of the appendix).

In this section we study the relationship between contiguity classes of simplicial maps and homotopy classes of the associated maps between finite spaces. These results appear in [11].

**Lemma 2.1.1.** Let \( f, g : X \to Y \) be two homotopic maps between finite \( T_0 \)-spaces. Then there exists a sequence \( f = f_0, f_1, \ldots, f_n = g \) such that for every \( 0 \leq i < n \) there is a point \( x_i \in X \) with the following properties:

1. \( f_i \) and \( f_{i+1} \) coincide in \( X \setminus \{x_i\} \), and
2. \( f_i(x_i) \prec f_{i+1}(x_i) \) or \( f_{i+1}(x_i) \prec f_i(x_i) \).

**Proof.** Without loss of generality, we may assume that \( f = f_0 \leq g \) by Corollary 1.2.6. Let \( A = \{ x \in X \mid f(x) \neq g(x) \} \). If \( A = \emptyset \), \( f = g \) and there is nothing to prove. Suppose \( A \neq \emptyset \) and let \( x = x_0 \) be a maximal point...
of $A$. Let $y \in Y$ be such that $f(x) < y \leq g(x)$ and define $f_1 : X \to Y$ by $f_1|_{X \setminus \{x\}} = f|_{X \setminus \{x\}}$ and $f_1(x) = y$. Then $f_1$ is continuous for if $x' > x$, $x' \notin A$ and therefore
\[ f_1(x') = f(x') = g(x') \geq g(x) \geq y = f_1(x). \]

Repeating this construction for $f_i$ and $g$, we define $f_{i+1}$. By finiteness of $X$ and $Y$ this process ends.

**Proposition 2.1.2.** Let $f, g : X \to Y$ be two homotopic maps between finite $T_0$-spaces. Then the simplicial maps $K(f), K(g) : K(X) \to K(Y)$ lie in the same contiguity class.

**Proof.** By the previous lemma, we can assume that there exists $x \in X$ such that $f(y) = g(y)$ for every $y \neq x$ and $f(x) < g(x)$. Therefore, if $C$ is a chain in $X$, $f(C) \cup g(C)$ is a chain on $Y$. In other words, if $\sigma \in K(X)$ is a simplex, $K(f)(\sigma) \cup K(g)(\sigma)$ is a simplex in $K(Y)$.

**Proposition 2.1.3.** Let $\varphi, \psi : K \to L$ be simplicial maps which lie in the same contiguity class. Then $\mathcal{X}(\varphi) \simeq \mathcal{X}(\psi)$.

**Proof.** Assume that $\varphi$ and $\psi$ are contiguous. Then the map $f : \mathcal{X}(K) \to \mathcal{X}(L)$, defined by $f(\sigma) = \varphi(\sigma) \cup \psi(\sigma)$ is well-defined and continuous. Moreover $\mathcal{X}(\varphi) \leq f \geq \mathcal{X}(\psi)$, and then $\mathcal{X}(\varphi) \simeq \mathcal{X}(\psi)$.

### 2.2 Minimal Pairs

In this section we generalize Stong’s ideas on homotopy types to the case of pairs $(X, A)$ of finite spaces (i.e. a finite space $X$ and a subspace $A \subseteq X$). As a consequence, we will deduce that every core of a finite $T_0$-space can be obtained by removing beat points from $X$. Here we introduce the notion of strong collapse which plays a central role in Chap. 5. Most of the results of this section appear in [11].

**Definition 2.2.1.** A pair $(X, A)$ of finite $T_0$-spaces is a minimal pair if all the beat points of $X$ are in $A$.

The next result generalizes the result of Stong (the case $A = \emptyset$) studied in Sect. 1.3 and its proof is very similar to the original one.

**Proposition 2.2.2.** Let $(X, A)$ be a minimal pair and let $f : X \to X$ be a map such that $f \simeq 1_X$ rel $A$. Then $f = 1_X$.

**Proof.** Suppose that $f \leq 1_X$ and $f|_A = 1_A$. Let $x \in X$. If $x \in X$ is minimal, $f(x) = x$. In general, suppose we have proved that $f|_{\hat{U}_x} = 1|_{\hat{U}_x}$, if $x \in A$, $f(x) = x$. If $x \notin A$, $x$ is not a down beat point of $X$. However $y < x$ implies $y = f(y) \leq f(x) \leq x$. Therefore $f(x) = x$. The case $f \geq 1_X$ is similar, and the general case follows from Corollary 1.2.6.