Path-dependent derivatives have payoffs that not only depend on the value $S_T$ of the underlying at maturity but also on the values \( \{S_t\}_{0 \leq t \leq T} \) attained up to maturity. They can broadly be classified into two categories: weakly and strongly path dependent. The former only depends on the value of the underlying at one or a few instances in time. These points in time might not be known today but can be determined by the future path taken by the underlying. An example is given by the derivative that pays the difference between the maximum value obtained in the time up to maturity and the value at maturity. The maximum only occurs at one instance in time, but viewed from today, we do not know when that will be. In contrast, a strongly path-dependent derivative depends on the whole path. An example is given by Asian options that have payoffs linked to the average of \( \{S_t\}_{0 \leq t \leq T} \) when monitored daily.

Weakly path-dependent derivatives can sometimes be priced with tools similar to those used for vanilla options. For instance, barrier options can be priced by adding boundary conditions to the same PDE used for vanillas. For strongly path-dependent derivatives, on the other hand, the pricing is fundamentally different. Indeed, it is often necessary to introduce variables that depend on the path. For instance, Asian options are usually modeled with the average value of the path as an extra variable. This leads to a PDE of one dimension higher.

Path-dependent derivatives are often priced numerically. The pricing is then done with a model calibrated to vanilla prices. If skew and smile effects are included, the calibration to vanilla instruments is typically done with perturbative methods or through the evaluation of low-dimensional integrals. The model is typically solved by simulating a SDE or by a numerical solution of a PDE. The reason for using numerical solutions for path-dependent derivatives is twofold: first of all, it is often hard to find suitable models for path-dependent options that can be solved by low-dimensional integrals, by simple perturbative techniques or other semi-analytical methods. Secondly, the types of path-dependent products that are popular change from client to client and from year to year. This is in contrast to vanilla products that are few and remain the same over the years. It is therefore useful to have generic
methods (by simulating SDEs or solving PDEs) for path-dependent derivatives rather than tailor-made methods.

We saw in Chap. 3 that the derivatives pricing problem can be formulated either in terms of SDEs or PDEs. From a theoretical point of view, these equivalent formulations complement each other: it is sometimes easier to analyze a pricing problem in terms of SDEs and sometimes in terms of PDEs. The same argument applies to numerical pricing. The main numerical difference between the two approaches is that SDEs are simulated forward in time while PDEs compute the price backwards in time from maturity to the pricing date. The PDE computations can be done either by using a finite difference approximation or through a tree structure using an explicit expression for the Green’s function. Several investment banks nowadays use a version of PDE solver consisting of recombining trees that are generalizations of binomial and trinomial trees with equally many nodes for each time slice. When going backwards in time and computing expectations, splines are used to connect the node points and the integrals can be solved by fast methods, usually with Gaussian quadrature. The high accuracy allows for large time steps with a substantial performance increase compared to traditional PDE solvers.

With experience, one learns for which derivatives SDEs are best suited and for which PDEs should be used. The basic guidelines are: PDEs usually perform better for low-dimensional problems, i.e. when there are only a small number of variables to track. Because of the technique of working backwards in time, PDE methods handle American and Bermudan features with ease. Furthermore, they return stable risk values. The SDE approach, on the other hand, is better performing in higher dimensions (typically equal to 3 or higher). Also, this approach is often simpler to implement and can be easily generalized to different types of payoffs.

There is a vast array of publications on the implementation of SDEs and PDEs, and we advise the interested reader to consult these. Path-dependent derivatives are sometimes also priced semi-analytically, but as there is currently no consensus on preferred methods, we have decided not to include these methods in the book. We instead focus on formulating the problems mathematically and solve them analytically if possible. The aim is not to provide state-of-the-art formulae but rather to help the reader to develop an intuition about path-dependent derivatives.

We use the constant-parameter lognormal SDE as an illustrating model. Please be aware of the fact that limiting ourselves to a lognormal model means that we do not have any control of the dynamics (which, as we pointed out in Sect. 4.3, is important for the pricing of path-dependent derivatives) or the possibility to calibrate to the skew and the smile. Furthermore, using time-independent parameters means that it is only possible to calibrate to a single maturity. Because of the limited space in this book, we only consider a small selection of products consisting of barrier options, variance swaps, American options and callable products. We believe that this set of products is large enough for the reader to gain familiarity with the techniques and to be able to price general path-dependent products.

Just as we did in Sect. 2.4 for European call options, it is possible to derive no-arbitrage conditions for path-dependent derivatives. For instance, a knock-out