11. The SK Model Without External Field

11.1 Overview

In this chapter we study the SK model without external field, with Hamiltonian

$$-H_N(\sigma) = \frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij} \sigma_i \sigma_j,$$  

(11.1)

for $\beta \leq 1$. This is the Hamiltonian (1.12) when $h = 0$. This model (at least for $\beta < 1$) is in some sense the simplest non trivial spin glass model, and not surprisingly more detailed results are available than for the more complicated cases. It enjoys some truly special features, one of which is that if $\beta < 1$ we have

$$E Z_N^2 \leq \frac{1}{\sqrt{1 - \beta^2}} (E Z_N)^2,$$  

(11.2)

where $Z_N$ is of course the partition function $\sum_\sigma \exp(-H_N(\sigma))$, and where the value of $\beta$ is kept implicit. A main difference between the cases $h \neq 0$ and $h = 0$ of the SK model (at high temperature) is that if $h \neq 0$ the fluctuations of $\log Z_N$ are typically of order $\sqrt{N}$, while if $h = 0$ they are typically of order 1. Consider the random variable

$$X = \log Z_N - N \left( \log 2 + \frac{\beta^2}{4} \right).$$

In Section 11.2, we prove exponential bounds for $\mathbb{P}(X \leq -t)$; and in Section 11.3 we prove exponential bounds for $\mathbb{P}(X \geq t)$. Not surprisingly, these bounds are obtained through specialized methods. In Section 11.4, we compute for each $k$ the limit $\lim_{N \to \infty} E X^k$, establishing a quantitative version of a central limit theorem of Aizenman, Lebowitz and Ruelle.

In Section 11.5 we examine the (random) matrix of the spin correlation $\langle \sigma_i \sigma_j \rangle$. We conjecture that this matrix shares some properties with the random matrix $(g_{ij}/\sqrt{N})$, and in particular that its operator norm remains bounded independently of $N$, a result that we prove within a logarithmic factor.

In Section 11.6 we examine some natural $d$-dimensional generalizations of the SK model without external field, for which we show that the high-
temperature phase extends much beyond that of the typical situation of Section 1.13.

The final Section 11.7 examines the case \( \beta = 1 \), and the case \( \beta = \beta_N \to 1 \) as \( N \to \infty \). This situation is replete with exciting problems, and our understanding is still very limited. Even such a basic question as determining the exact order of \( \nu(R_{1,2}^2) \) when \( \beta = 1 \) is wide open (and looks very difficult).

This results of this chapter are largely independent from those of Chapter 1, but the reader should be at least be familiar with Section 16.

11.2 Lower Deviations for \( Z_N \)

The goal of the section is to prove the following:

**Theorem 11.2.1.** Given \( \beta < 1 \), there exists \( K \) depending on \( \beta \) only such that for any \( N \) and any \( t > 0 \):

\[
P \left( \log Z_N \leq N \left( \frac{\beta^2}{4} + \log 2 \right) - t \right) \leq K \exp \left( -\frac{t^2}{K} \right).
\]

This deviation inequality raises the following “large deviation” problem.

**Research Problem 11.2.2.** (Level 2) Given \( \beta < 1 \) and \( t > 0 \), prove the existence of the limit

\[
\lim_{N \to \infty} \frac{1}{N^2} \log P \left( \frac{1}{N} \log Z_N \leq \frac{\beta^2}{4} + \log 2 - t \right)
\]

and compute it. More generally, given \( 0 \leq \alpha < 1 \), compute the limit

\[
\lim_{N \to \infty} \frac{1}{N^{2(1-\alpha)}} \log P \left( \frac{1}{N} \log Z_N \leq \frac{\beta^2}{4} + \log 2 - \frac{t}{N^\alpha} \right).
\]

We now prepare for the proof of Theorem 11.2.1. The fundamental relation (11.2) is the special case \( \gamma = 0 \) of the following (which will also be useful in its own right):

**Lemma 11.2.3.** If \( \gamma + \beta^2 < 1 \) we have

\[
E \sum_{\sigma^1, \sigma^2} \exp \left( -H_N(\sigma^1) - H_N(\sigma^2) + \frac{\gamma N}{2} R_{1,2}^2 \right) \leq \frac{1}{\sqrt{1 - \beta^2 - \gamma}} (EZ_N)^2.
\]

(11.3)