14. The Parisi Formula

14.1 Introduction

In this chapter we obtain a considerable extension of the results of Chapter 13: we compute in the limit the quantity \( p_N(\beta, h) \) for any values of \( \beta \) and \( h \). The result, called the Parisi formula, is very beautiful but is not immediate to explain, and its proof is rather involved. A large part of the motivation of Chapter 13 is to present a simpler special case of the main arguments of the present chapter, and we advise the reader to first master the first six sections of Chapter 13 before attempting to read anything at all here. Starting with Section 14.7 matters get a bit technical, but let us insist that the reader certainly does not need to master the details of the computations in order to enjoy the next chapter, which attempts to describe the structure underlying the Parisi formula.

14.2 Poisson-Dirichlet Cascades

In this section we briefly discuss certain objects that we call Poisson-Dirichlet cascades, which seem intrinsically connected to the low-temperature phase of spin-glass systems. These objects are very pretty, but we will refrain from the temptation of studying them for their own sake, and will present only the results that are really helpful in the sequel.

Consider an integer \( k \) and the set \( \mathbb{N}^* = \{1, 2, \ldots, \} \). We will denote by \( \alpha \) a sequence \((j_1, \ldots, j_k)\) in \( \mathbb{N}^*^k \). It could be useful to think of \( \mathbb{N}^*^k \) as a tree. As we scan the integers \( j_1, j_2, \ldots \) we discover which branch we follow at each node. For \( p \leq k \) we write \( \alpha \mid p = (j_1, \ldots, j_p) \in \mathbb{N}^*^p \). Let us fix a sequence \( 0 < m_1 < m_2 < \ldots < m_k < 1 \). Given this sequence we are going to construct random quantities \((u^*_\alpha)\), \( \alpha \in \mathbb{N}^*^k \). Let us first consider a non-increasing rearrangement \((u_{j_1})_{j \geq 1}\) of a Poisson point process of intensity measure \( x^{-m_1-1}dx \). For each integer \( j_1 \), we consider a non-increasing rearrangement \((u_{j_1, j})_{j \geq 1}\) of a Poisson point process of intensity measure \( x^{-m_2-1}dx \). These are all independent of each other and of the sequence \((u_j)\). More generally, for each \( 1 \leq p \leq k \) and each integers \( j_1, \ldots, j_{p-1} \) we consider a non-increasing rearrangement \((u_{j_1, \ldots, j_{p-1}, j})_{j \geq 1}\) of a Poisson point process of intensity measure \( x^{-m_p-1}dx \). All these are independent of each other.
\[ u^*_\alpha = u_{\alpha|1} u_{\alpha|2} \cdots u_{\alpha|k-1} u_{\alpha} \]
\[ = u_{j_1} u_{j_1 j_2} \cdots u_{j_1 \cdots j_k} . \]  

(14.1)

It will be shown in (14.9) below (taking \( F = 0 \) there) that \( \sum_\gamma u^*_\gamma < \infty \) a.s.

The family of weights \( (v_\alpha)_{\alpha \in \mathbb{N}^k} \) where

\[ v_\alpha = \frac{u^*_\alpha}{\sum u^*_\gamma} \]

(14.2)

will be called the *Poisson-Dirichlet cascade associated with the sequence* \( m_1, \ldots, m_k \). When \( k = 1 \), the sequence \( v_\alpha \) has the Poisson-Dirichlet distribution \( \Lambda_{m_1} \) first defined in Section 13.1 page 314.

In this chapter and the next, we will often consider random weights. Weights associated to a Poisson-Dirichlet cascade will be denoted by \( (v_\alpha) \) while we will denote by \( (w_\alpha) \) weights that need not be of this type.

Before we can describe some remarkable properties of this object, we must explain another procedure that will be fundamental, maybe even more so than Poisson-Dirichlet cascades. Consider a metric space \( T \). (The case \( T = \mathbb{R}^N \) or \( T = \mathbb{R} \) will be the most useful.) Consider a function \( F : T^k \to \mathbb{R} \). Consider also independent random maps \( z_1, \ldots, z_k \) valued in \( T \), and define the r.v.

\[ F_{k+1} = F(z_1, \ldots, z_k) . \]  

(14.3)

We will assume that

\[ \mathbb{E} \exp F_{k+1} < \infty ; \mathbb{E}|F_{k+1}| < \infty . \]

(14.4)

For \( 1 \leq p \leq k \), let us denote by \( \mathbb{E}_p \) expectation in the r.v.s \( z_p, \ldots, z_k \), and let us define recursively the r.v.s

\[ F_p = \frac{1}{m_p} \log \mathbb{E}_p \exp m_p F_{p+1} , \]

(14.5)

so that \( F_p \) depends only on the r.v.s \( z_1, \ldots, z_{p-1} \), and in particular \( F_1 \) is a number.

For each \( p \leq k \) and integers \( j_1, \ldots, j_p \), let us consider independent copies \( z_{p,j_1, \ldots, j_p} \) of \( z_p \). These are all independent of each other. The reader should observe the similarity with the procedure by which we define the r.v.s \( u^*_\alpha \). To lighten notation for \( \alpha = (j_1, \ldots, j_k) \in \mathbb{N}^k \) we write

\[ z_{p,\alpha} = z_{p,j_1, \ldots, j_p} . \]

(14.6)

This variable depends only on \( p \) and \( \alpha|p \). The procedure of defining the r.v.s \( z_{p,\alpha} \) from the r.v.s \( z_p \) will occur a great many times, and the notation (14.6) remains in force through the chapter.

Let us then define

\[ F(\alpha) = F(z_{1,\alpha}, \ldots, z_{k,\alpha}) . \]

(14.7)