Chapter 5
Autonomous Single Oscillator

This chapter studies finite amplitude vibrations of the autonomous mechanical systems having one degree of freedom. The character of solutions depends strongly on the type of the system. The solution methods may range from phase portrait and Lindstedt-Poincaré method for conservative systems up to Bogoliubov-Mitropolsky method for systems with weak dissipation.

5.1 Conservative Oscillator

Differential equation of motion. As before, Hamilton’s variational principle with the Lagrange function \( L(q, \dot{q}) \), \( q \) and \( \dot{q} \) being the generalized coordinate and velocity, is our tool for deriving the equation of motion of conservative systems. However, in contrast to the linear theory, we will see that the kinetic energy may now depend on \( q \) as well, and the potential energy is no longer quadratic with respect to \( q \). We consider three simple examples.

Example 5.1 Mass-spring oscillator. A point-mass \( m \) moves horizontally under the action of a non-linear spring (see Fig. 5.1). Derive the equation of motion for this oscillator.

Like the oscillator considered in example [1.1] the kinetic energy is given by \( K = \frac{1}{2} m\dot{x}^2 \). Concerning the potential energy of the non-linear spring we first consider the most general case, for which \( U(x) \) is an arbitrary smooth function. Then Lagrange’s equation reads

\[
mx - f(x) = 0, \quad f(x) = -\frac{dU}{dx}.
\]

![Fig. 5.1 Mass-spring oscillator.](image-url)
The spring force \( f(x) \) is called a restoring force. However, it is quite reasonable to assume that the potential energy of the spring deviates only slightly from that of the linear spring, i.e.,

\[
U(x) = \frac{1}{2} k x^2 + \frac{1}{4} \alpha \frac{k}{l_0^2} x^4,
\]

where \( l_0 \) is the original length of the spring and \( \alpha \) a small parameter. If \( \alpha > 0 \), the spring is called hardening; on the contrary if \( \alpha < 0 \) it is called softening. Lagrange’s equation becomes

\[
m\ddot{x} + kx + \alpha \frac{k}{l_0^2} x^3 = 0.
\]

Dividing this equation by \( kl_0 \) and rewriting it in terms of the dimensionless function \( \bar{x} = x/l_0 \) and the dimensionless time \( \bar{t} = \sqrt{k/m} t \) we obtain

\[
\ddot{x} + x + \alpha x^3 = 0. \tag{5.1}
\]

Equation (5.1) is known as Duffing’s equation.

**EXAMPLE 5.2** Derive the equation of motion of the mathematical pendulum considered in example [1.2].

As has been shown already in that example, the Lagrange function is

\[
L(\varphi, \dot{\varphi}) = \frac{1}{2} ml^2 \dot{\varphi}^2 - mgl(1 - \cos \varphi),
\]

but now \( \varphi \) is no longer small. Thus, the finite amplitude vibrations of this pendulum are described by the equation

\[
\ddot{\varphi} + \omega_0^2 \sin \varphi = 0, \quad \omega_0 = \sqrt{\frac{g}{l}}.
\]

By expanding \( \sin \varphi \) in the Taylor series about \( \varphi = 0 \) and keeping the terms up to \( \varphi^3 \) we obtain the approximate equation

\[
\ddot{\varphi} + \omega_0^2 (\varphi - \frac{\varphi^3}{6}) = 0,
\]

which can be transformed to (5.1) with \( \alpha = -1/6 \).

**EXAMPLE 5.3** A point-mass \( m \) is constrained to move along a frictionless path represented by a smooth curve \( y = y(x) \) in the \((x,y)\)-plane under the action of gravity (see Fig. 5.2). Derive the equation of motion.

\[1\] The bar is dropped for short.