Chapter 9
Dynamical System Interactions

In this chapter, discontinuous dynamical system theory will be applied to dynamical system interactions. The concept of interaction between two dynamical systems will be introduced. An interaction condition of two dynamical systems will be treated as a separation boundary, and such a boundary is time-varying. In other words, the boundary and domains for one of two dynamical systems are constrained by the other. The corresponding conditions for such an interaction will be presented via the theory for the switchability and attractivity of edge flows to the specific edges. The synchronization of two totally different dynamical systems will be presented as an application.

9.1. Introduction to system interactions

In this section, basic concepts of the dynamical system interactions will be presented. The discontinuous description of the interaction of two dynamical systems will be presented.

9.1.1. System interactions

Definition 9.1. Two dynamical systems are defined by
\[
\dot{y} = F(y, t, p) \in \mathbb{R}^n \quad \text{and} \quad \dot{x} = \mathcal{F}(x, t, q) \in \mathbb{R}^m. \tag{9.1}
\]
If two flows \( x(t) \) and \( y(t) \) of the two systems in Eq. (9.1) satisfy
\[
\varphi(x(t), y(t), t, \lambda) = 0, \quad \lambda \in \mathbb{R}^n, \tag{9.2}
\]
then the two systems are called to be interacted (or constrained) under such a condition at time \( t \).
From the foregoing definition, the interaction (or constraint) of two dynamical systems in Eq. (9.1) occurs through \( \varphi(x(t), y(t), t, \lambda) = 0 \) in Eq. (9.2). Such a condition may cause the discontinuity for two dynamical systems. If the interaction condition is the separation boundary, then the domain and boundary for the first dynamical system in Eq. (9.1) will be time-varying, which is controlled by a flow of the second dynamical system in Eq. (9.1) (i.e., \( x(t) \)), vice versa. Suppose the interaction of two systems occurs at time \( t \). For time \( t \pm \varepsilon \ (\varepsilon > 0) \), there are two constants with

\[
\varphi(x, y, t \pm \varepsilon, \lambda) = C_{\pm} \neq 0.
\]

If the flows of two systems in Eq. (9.1) satisfy Eq. (9.3), then the two systems will not be interacted, as shown in Fig. 9.1. In fact, the interaction of two dynamical systems can occur under many constraints instead of Eq. (9.2), i.e.,

**Definition 9.2.** Consider \( l \)-non-identical functions of \( \varphi_j(x(t), y(t), t, \lambda_j) \) ( \( j \in \mathcal{L} \) and \( \mathcal{L} = \{1, 2, \cdots, l\} \)). If two flows \( x(t) \) and \( y(t) \) of two systems in Eq. (9.1) satisfy for time \( t \)

\[
\varphi_j(x(t), y(t), t, \lambda_j) = 0 \quad \text{for} \quad \lambda_j \in \mathbb{R}^{n_j} \quad \text{and} \quad j \in \mathcal{L}.
\]

then two systems in Eq. (9.1) are called to be constrained under the \( l \)-condition at time \( t \).

For the foregoing definition, two dynamical systems in Eqs. (9.1) possess \( l \)-conditions for interactions (or constraints). Thus, the \( l \)-separation boundaries rela-