Chapter 4

Singular Lane–Emden–Fowler Equations and Systems

4.1 Bifurcation Problems for Singular Elliptic Equations

In this section we study the bifurcation problem

\[
\begin{cases}
-\Delta u = \lambda f(u) + a(x)g(u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \lambda \in \mathbb{R} \) is a parameter and \( \Omega \subset \mathbb{R}^N \) \((N \geq 2)\) is a bounded domain with smooth boundary \( \partial \Omega \). The main feature of this boundary value problem is the presence of the “smooth” nonlinearity \( f \) combined with the “singular” nonlinearity \( g \). More exactly, we assume that \( 0 < f \in C^{0, \beta}[0, \infty) \) and \( 0 \leq g \in C^{0, \beta}(0, \infty) \) \((0 < \beta < 1)\) fulfill the hypotheses

\begin{enumerate}
\item[(f1)] \( f \) is nondecreasing on \((0, \infty)\) while \( f(s)/s \) is nonincreasing for \( s > 0 \).
\item[(g1)] \( g \) is nonincreasing on \((0, \infty)\) with \( \lim_{s \downarrow 0} g(s) = +\infty \).
\item[(g2)] there exist \( C_0, \eta_0 > 0 \) and \( \alpha \in (0, 1) \) so that \( g(s) \leq C_0 s^{-\alpha}, \forall s \in (0, \eta_0) \).
\end{enumerate}
The assumption \(g_2\) implies the following Keller–Osserman-type growth condition around the origin

\[
\int_0^1 \left( \int_0^t g(s) ds \right)^{-1/2} dt < +\infty. \tag{4.1}
\]

As proved by Bénilan, Brezis and Crandall in [14], condition (4.1) is equivalent to the property of compact support, that is, for any \(h \in L^1(\mathbb{R}^N)\) with compact support, there exists a unique \(u \in W^{1,1}(\mathbb{R}^N)\) with compact support such that \(\Delta u \in L^1(\mathbb{R}^N)\) and

\[-\Delta u + g(u) = h \quad \text{a.e. in } \mathbb{R}^N.\]

In many papers (see, e.g., Dalmasso [56], Kusano and Swanson [125]) the potential \(a(x)\) is assumed to depend “almost” radially on \(x\), in the sense that \(C_1 p(|x|) \leq a(x) \leq C_2 p(|x|)\), where \(C_1, C_2\) are positive constants and \(p(|x|)\) is a positive function satisfying some integrability condition. We do not impose any growth assumption on \(a\), but we suppose that the variable potential \(a(x)\) satisfies \(a \in C^{0,\beta}(\overline{\Omega})\) and \(a > 0\) in \(\Omega\).

If \(\lambda = 0\) this equation is called the Lane–Emden–Fowler equation and arises in the boundary-layer theory of viscous fluids (see Wong [213]). Problems of this type, as well as the associated evolution equations, describe naturally certain physical phenomena. For example, super-diffusivity equations of this type have been proposed by de Gennes [62] as a model for long range Van der Waals interactions in thin films spreading on solid surfaces.

Our purpose is to study the effect of the asymptotically linear perturbation \(f(u)\) in \((P_\lambda)\), as well as to describe the set of values of the positive parameter \(\lambda\) such that problem \((P_\lambda)\) admits a solution. In this case, we also prove a uniqueness result. Due to the singular character of \((P_\lambda)\), we can not expect to find solutions in \(C^2(\overline{\Omega})\). However, under the above assumptions we will show that \((P_\lambda)\) has solutions in the class

\[\mathcal{E} := \{ u \in C^2(\Omega) \cap C^{1,1-\alpha}(\overline{\Omega}); \Delta u \in L^1(\Omega)\}.\]

We first observe that, in view of the assumption \((f1)\), there exists

\[m := \lim_{s \to +\infty} \frac{f(s)}{s} \in [0, \infty).\]

This number plays a crucial role in our analysis. More precisely, the existence of the solutions to \((P_\lambda)\) will be separately discussed for \(m > 0\) and \(m = 0\). Let \(a_* = \min_{x \in \Omega} a(x)\).