Chapter 4

PDEs with Diffusion

The aim of this chapter is to investigate the approximation of scalar PDEs with diffusion. As a first step, we consider the Poisson problem with homogeneous Dirichlet boundary condition

\[-\Delta u = f \quad \text{in } \Omega, \tag{4.1a}\]
\[u = 0 \quad \text{on } \partial \Omega, \tag{4.1b}\]

and source term \(f \in L^2(\Omega)\). The scalar-valued function \(u\) is termed the potential and the vector-valued function \(-\nabla u\) the diffusive flux.

In Sect. 4.1, we briefly describe the continuous setting for the model problem (4.1). Then, in the following sections, we discuss three possible approaches to design a dG approximation for this problem. In Sect. 4.2, we present a heuristic derivation of a suitable discrete bilinear form loosely following the same path of ideas as in Chap. 2 hinging on consistency and discrete coercivity. There are, however, substantial differences: a specific term needs to be added to recover consistency, interface jumps as well as boundary values are penalized, and the penalty term scales as the reciprocal of the local meshsize so that discrete coercivity is expressed using a mesh-dependent norm. A further important difference is that we require to work at least with piecewise affine polynomials, thereby excluding, for the time being, methods of finite volume-type. This derivation yields the so-called Symmetric Interior Penalty (SIP) method of Arnold [14].

The error analysis follows fairly standard arguments and leads to optimal error estimates for smooth exact solutions. We also present a more recent analysis by the authors [132] in the case of low-regularity exact solutions. Then, using liftings of the interface jumps and boundary values, we introduce in Sect. 4.3 the important concept of discrete gradient. Applications include (1) a reformulation of the SIP bilinear form that plays a central role in Sect. 5.2 to analyze the convergence to minimal regularity solutions (as shown recently by the authors in [131]), and (2) an elementwise formulation of the discrete problem leading to a local conservation property in terms of numerical fluxes. Finally, the third approach is pursued in Sect. 4.4 where we consider mixed dG methods that approximate...
both the potential and the diffusive flux. In such methods, local problems for the discrete potential and diffusive flux are formulated using numerical fluxes for both quantities, following the pioneering works of Bassi, Rebay, and coworkers [34, 35] and Cockburn and Shu [112]. This viewpoint has been adopted by Arnold, Brezzi, Cockburn, and Marini in [16] for a unified presentation of dG methods for the Poisson problem. For simplicity, we focus here on the SIP method and so-called Local Discontinuous Galerkin (LDG) methods [112]. In both methods, the discrete diffusive flux can be eliminated locally. We postpone the study of two-field dG methods to Sect. 7.3 in the more general context of Friedrichs’ systems. Finally, we discuss hybrid mixed dG methods where additional degrees of freedom are introduced at interfaces, thereby allowing one to eliminate locally both the discrete potential and the discrete diffusive flux. This leads, in particular, to the so-called Hybridized Discontinuous Galerkin (HDG) methods introduced by Cockburn, Gopalakrishnan, and Lazarov [97]; see also Causin and Sacco [83] for a different approach based on a discontinuous Petrov–Galerkin formulation and Droniou and Eymard [135] for similar ideas in the context of hybrid mixed finite volume schemes.

The rest of this chapter is devoted to the study of diffusive PDEs that comprise additional terms with respect to the model problem (4.1). In Sect. 4.5, we extend the SIP method analyzed in Sect. 4.2 to heterogeneous (anisotropic) diffusion problems. The main ingredients are diffusion-dependent weighted averages to formulate the consistency and symmetry terms in the discrete bilinear form together with diffusion-dependent penalty parameters using the harmonic mean of the diffusion coefficient at each interface. In Sect. 4.6, we analyze heterogeneous diffusion-advection-reaction problems. We combine the ideas of Sect. 4.5 to handle the diffusion term with those developed in Sect. 2.3 for the upwind dG method to handle the advection-reaction term. The goal is a convergence analysis that covers both diffusion-dominated and advection-dominated regimes. The present analysis includes the case where the diffusion vanishes locally in some parts of the domain. Finally, in Sect. 4.7, we consider the heat equation as a prototype for time-dependent scalar PDEs with diffusion (that is, parabolic PDEs). The approximation is based on the SIP method for space discretization and an A-stable finite difference scheme in time; for simplicity, we focus on backward (or implicit) Euler and BDF2 schemes.

4.1 Pure Diffusion: The Continuous Setting

In this section, we present some basic facts concerning the model problem (4.1).

4.1.1 Weak Formulation and Well-Posedness

The weak formulation of (4.1) is classical:

\[ \text{Find } u \in V \text{ s.t. } a(u, v) = \int_{\Omega} fv \text{ for all } v \in V, \] (4.2)