27 A computational framework for nonlinear elasticity

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Nonlinear elasticity theory plays a fundamental role in modeling the mechanical response of many polymeric and biological materials. Such materials are capable of undergoing finite deformation, and their material response is often characterized by complex, nonlinear constitutive relationships. (See, for example, Holzapfel (2000) and Truesdell and Noll (1965) and the references within for several examples.) Because of these difficulties, predicting the response of arbitrary structures composed of such materials to arbitrary loads requires numerical computation, usually based on the finite element method. The steps involved in the construction of the required finite element algorithms are classical and straightforward in principle, but their application to non-trivial material models are typically tedious and error-prone. Our recent work on an automated computational framework for nonlinear elasticity, CBC.Twist, is an attempt to alleviate this problem.

The focus of this chapter will be to describe the design and implementation of CBC.Twist, as well as providing examples of its use. The goal is to allow researchers to easily pose and solve problems in nonlinear elasticity in a straightforward manner, so that they may focus on higher-level modeling questions without being hindered by specific implementation issues.

What follows is the proposed outline for the chapter.

The chapter begins with a summary of some key results from classical nonlinear elasticity theory. This discussion is used to motivate the design of CBC.Twist, which is a DOLFIN (Logg and Wells, 2010) module written in UFL syntax (Alnæs and Logg, 2009) that closely resembles how the theory is written down on paper. In particular, we will see how one can easily pose sophisticated material models purely at the level of specifying a strain energy function. The discourse will then turn to the primary equation of interest: the balance of linear momentum of a body posed in the reference configuration. A finite element scheme for this equation will then be presented, pointing out how CBC.Twist leverages the automatic linearization capabilities of UFL to implement this scheme in a manner that is independent of the specific material model. The time-stepping schemes that CBC.Twist implements will also be discussed. With this in place, we turn to increasingly complex examples to see how initial- boundary-value problems in nonlinear elasticity can be posed and solved in CBC.Twist using only a few lines of high-level code. The chapter concludes with some remarks on how one can obtain CBC.Twist, along with ideas for its extension.
27.1 Brief overview of nonlinear elasticity theory

The goal of this section is to present an overview of the mathematical theory of nonlinear elasticity, which plays an important role in the design of CBC.Twist. Readers interested in a more comprehensive treatment of the subject are referred to, for example, the classical treatises of Truesdell and Toupin (1960) and Truesdell and Noll (1965), or more modern works such as Gurtin (1981), Ogden (1997) and Holzapfel (2000).

27.1.1 Posing the question we aim to answer

The theory begins by idealizing the elastic body of interest as an open subset of \( \mathbb{R}^{2,3} \) with a piecewise smooth boundary. At a reference placement of the body, \( \Omega \), points in the body are identified by their reference positions, \( X \in \Omega \). The treatment presented in this chapter is posed in terms of fields which are parametrized by reference positions. This is commonly termed the material or Lagrangian description.

In its most basic terms, the deformation of the body over time \( t \in [0, T] \) is a sufficiently smooth bijective map \( \varphi : \overline{\Omega} \times [0, T] \to \mathbb{R}^{2,3} \), where \( \overline{\Omega} := \Omega \cup \partial \Omega \) and \( \partial \Omega \) is the boundary of \( \Omega \). The restrictions on the map ensure that the motion it describes is physical and within the range of applicability of the theory (e.g., disallowing the interpenetration of matter or the formation of cracks). From this map, we can construct the displacement field,

\[
    u(X, t) = \varphi(X, t) - X, \tag{27.1}
\]

which represents the displacement of a point in time relative to its reference position.

With this brief background, we are ready to pose the fundamental question that CBC.Twist is designed to answer: Given a body comprised of a specified elastic material, what is the displacement of the body when it is subjected to prescribed:

- **Body forces:** These include forces such as the self-weight of a body, forces on ferromagnetic materials in magnetic fields, etc., which act everywhere in the volume of the body. They are denoted by the vector field \( B(X, t) \).

- **Traction forces:** This is the force measured per unit surface area acting on the Neumann boundary of the body, \( \partial \Omega_N \), and denoted by the vector field \( T(X, t) \).

- **Displacement boundary conditions:** These are displacement fields prescribed on the Dirichlet boundary of the body, \( \partial \Omega_D \).

It is assumed that \( \partial \Omega_N \cap \partial \Omega_D = \emptyset \) and \( \overline{\partial \Omega_N} \cup \overline{\partial \Omega_D} = \partial \Omega \). These details are depicted in Figure 27.1.

Figure 27.1: An elastic body idealized as a continuum, subjected to body forces, \( B \), surface tractions, \( T \), and prescribed displacement boundary conditions.