5 Finite element variational forms

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Much of the FEniCS software is devoted to the formulation of variational forms (UFL), the discretization of variational forms (FIAT, FFC, SyFi) and the assembly of the corresponding discrete operators (UFC, DOLFIN). This chapter summarizes the notation for variational forms used throughout FEniCS.

5.1 Background

In Chapter 2, we introduced the following canonical variational problem: find \( u \in V \) such that
\[
a(u, v) = L(v) \quad \forall v \in \hat{V},
\]
where \( V \) is a given trial space and \( \hat{V} \) is a given test space. The bilinear form
\[
a : V \times \hat{V} \to \mathbb{R}
\]
maps a pair of trial and test functions to a real number and is linear in both arguments. Similarly, the linear form \( L : \hat{V} \to \mathbb{R} \) maps a given test function to a real number. We also considered the discretization of nonlinear variational problems: find \( u \in V \) such that
\[
F(u; v) = 0 \quad \forall v \in \hat{V}.
\]
Here, \( F : V \times \hat{V} \to \mathbb{R} \) again maps a pair of functions to a real number. The semilinear form \( F \) is nonlinear in the function \( u \) but linear in the test function \( v \). Alternatively, we may consider the mapping
\[
L_u \equiv F(u; \cdot) : \hat{V} \to \mathbb{R},
\]
and note that \( L_u \) is a linear form on \( \hat{V} \) for any fixed value of \( u \). In Chapter 2, we also considered the estimation of the error in a given functional \( M : V \to \mathbb{R} \). Here, the possibly nonlinear functional \( M \) maps a given function \( u \) to a real number \( M(u) \).

In all these examples, the central concept is that of a form that maps a given tuple of functions to a real number. We shall refer to these as multilinear forms. Below, we formalize the concept of a multilinear form, discuss the discretization of multilinear forms, and related concepts such as the action, derivative and adjoint of a multilinear form.
5.2 Multilinear forms

A form is a mapping from the product of a given sequence \( \{V_j\}_{j=1}^{\rho} \) of function spaces to a real number:

\[
a : V_\rho \times \cdots \times V_2 \times V_1 \to \mathbb{R}.
\]  

(5.5)

If the form \( a \) is linear in each of its arguments, we say that the form is multilinear. The number of arguments \( \rho \) of the form is the arity of the form. Note that the spaces are numbered from right to left. As we shall see below in Section 5.3, this is practical when we consider the discretization of multilinear forms.

Forms may often be parametrized over one or more coefficients. A typical example is the right-hand side \( L \) of the canonical variational problem (5.1), which is a linear form parametrized over a given coefficient \( f \). We shall use the notation \( a(f; v) \equiv L_f(v) \equiv L(v) \) and refer to the test function \( v \) as an argument and to the function \( f \) as a coefficient. In general, we shall refer to forms which are linear in each argument (but possibly nonlinear in its coefficients) as multilinear forms. Such a multilinear form is a mapping from the product of a sequence of argument spaces and coefficient spaces:

\[
a : W_1 \times W_2 \times \cdots \times W_n \times V_\rho \times \cdots \times V_2 \times V_1 \to \mathbb{R},
\]

\[
a \mapsto a(w_1, w_2, \ldots, w_n; v_\rho, \ldots, v_2, v_1).
\]

(5.6)

The argument spaces \( \{V_j\}_{j=1}^{\rho} \) and coefficient spaces \( \{W_j\}_{j=1}^n \) may all be the same space but they typically differ, such as when Dirichlet boundary conditions are imposed on one or more of the spaces, or when the multilinear form arises from the discretization of a mixed problem such as in Section 2.2.2.

In finite element applications, the arity of a form is typically \( \rho = 2 \), in which case the form is said to be bilinear, or \( \rho = 1 \), in which case the form is said to be linear. In the special case of \( \rho = 0 \), we shall refer to the multilinear form as a functional. It may sometimes also be of interest to consider forms of higher arity (\( \rho > 2 \)). Below, we give examples of some multilinear forms of different arity.

5.2.1 Examples

Poisson’s equation. Consider Poisson’s equation with variable conductivity \( \kappa = \kappa(x) \),

\[
- \text{div}(\kappa \text{grad} u) = f.
\]  

(5.7)

Assuming Dirichlet boundary conditions on the boundary \( \partial \Omega \), the corresponding canonical variational problem is defined in terms of a pair of multilinear forms, \( a(\kappa; u, v) = \int_\Omega \kappa \text{grad} u \cdot \text{grad} v \, dx \) and \( L(v) = \int_\Omega fv \, dx \). Here, \( a \) is a bilinear form (\( \rho = 2 \)) and \( L \) is a linear form (\( \rho = 1 \)). Both forms have one coefficient (\( n = 1 \)) and the coefficients are \( \kappa \) and \( f \) respectively:

\[
a = a(\kappa; u, v),
\]

\[
L = L(f; v).
\]

(5.8)

We usually drop the coefficients from the notation and use the short-hand notation \( a = a(u, v) \) and \( L = L(v) \).

The incompressible Navier–Stokes equations. The incompressible Navier–Stokes equations for the velocity \( u \) and pressure \( p \) of an incompressible fluid read:

\[
\rho(\dot{u} + \text{grad} u \cdot u) - \text{div} \sigma(u, p) = f,
\]

\[
\text{div} u = 0,
\]

(5.9)