Chapter 7
Model Equations and Methods of Finding Their Exact Solutions

7.1 Introduction

The first part of this book has a very general character. It is devoted to nonlinear waves and wave structures (such as shock fronts, solitary waves, cellular multidimensional structures and some others) considered from the position of the general nonlinear wave theory. This approach is especially important for familiarization with the material of a big area of nonlinear physics — the theory of propagation of nondispersive and weakly dispersive waves, and also related to it fields of science united by similar mathematical models and methods of their analysis. Hence Part I may be used for educational purposes.

However, along with the general point of view, it is of interest to turn to an expanded treatment of the models and ideas of Part I as applied to specific physical situations. As it is known, one of the most important (from the viewpoint of applications) areas of physics of waves in nondispersive media is nonlinear acoustics or physics of powerful acoustic fields [1–4]. A detailed description of the relevant theory and particularization of the material from Part I as applied to nonlinear acoustics are given in Part II of this book.

7.1.1 Facts from the linear theory

Let us begin with the general information on the linear theory of acoustic waves [2, 4], which is necessary for the consequent analysis of nonlinear problems.

The system of equations describing motion of liquids and gases, while taking their shear $\eta$ and bulk $\varsigma$ viscosities into account, consists of the equation of motion in the Navier-Stokes form

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \mathbf{u} \right] = -\nabla p + \eta \Delta \mathbf{u} + \left( \varsigma + \frac{\eta}{3} \right) \text{grad div} \, \mathbf{u}, \quad (7.1)$$

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the continuity equation
\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0 \]  
(7.2)

and the equation of state, which, for acoustic waves, may be taken in the form of the Poisson adiabat
\[ p = p(\rho) = p_0 \left( \frac{\rho}{\rho_0} \right)^\gamma. \]  
(7.3)

In writing the system (7.1)–(7.3), we used Euler’s approach towards a description of continuous medium motion. Within the framework of this approach, all variables — pressure \( p \), density \( \rho \) and velocity \( \mathbf{u} \) — are considered to be functions of coordinates in a stationary reference frame, as well as of time \( t \). Often, especially in one-dimensional problems, Lagrange’s approach is used. Within the Lagrangian description of continuous medium motion, instead of the coordinates of a stationary reference frame, coordinates of liquid particles of the medium, measured at a certain fixed (initial) moment of time, are used. Lagrange’s approach is the main one in the solid state elasticity theory, when relative displacements of different inner volume domains of a body, responsible for appearance of mechanical stresses and deformations, and not the translations of the body as a whole, are of interest.

Let the unperturbed state of a medium be \( \rho = \rho_0, p = p_0, \mathbf{u} = 0 \). Let us denote perturbations of parameters connected with a wave by primed letters and assign in the system (7.1)–(7.3)
\[ p = p_0 + p', \quad \rho = \rho_0 + \rho'. \]  
(7.4)

We consider the perturbation to be small:
\[ \frac{p'}{p_0} \sim \frac{\rho'}{\rho_0} \sim \frac{|\mathbf{u}|}{c_0} \sim \mu \ll 1. \]  
(7.5)

Here \( \mu \) is the small parameter, whose powers will be used in order to expand equations into series,
\[ c_0 = \sqrt{\left( \frac{\partial p}{\partial \rho} \right)_{\rho=p_0}} = \sqrt{\frac{\gamma p_0}{\rho_0}} \]  
(7.6)
is the equilibrium speed of sound. The ratio of the velocity of liquid’s particle motion to the sound speed \( |\mathbf{u}|/c_0 \) is called the acoustic Mach number \( M \). From Eq. (7.5), it is seen that the small parameter \( \mu \) has the same order of smallness as the acoustic Mach number.

By substituting (7.4) into the system of Eqs. (7.1)–(7.3), it is reduced to the following form:
\[ \rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p' - \eta \Delta \mathbf{u} - \left( \zeta + \frac{\eta}{3} \right) \text{grad} \mathbf{u} = -\rho' \frac{\partial \mathbf{u}}{\partial t} - (\rho_0 + \rho') (\mathbf{u} \nabla) \mathbf{u}, \]  
(7.7)
\[ \frac{\partial \rho'}{\partial t} + \rho_0 \text{div} \mathbf{u} = -\text{div}(\rho' \mathbf{u}), \]  
(7.8)
\[ p' = c_0^2 \rho' + \frac{1}{2} \left( \frac{\partial^2 p}{\partial \rho^2} \right) \rho'^2 + \cdots \equiv C_1 \frac{\rho'}{\rho_0} + \frac{1}{2} C_2 \left( \frac{\rho'}{\rho_0} \right)^2 + \cdots \]  
(7.9)