9.1 Cylinders of Plane Curve Singularities

Consider \( f(x, y, z) = f'(x, y) \) and \( g(x, y, z) = z \), where \( f' : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) is an isolated plane curve singularity. It is well-known (see e.g. [16, 45, 136]) that the embedded resolution of \( (\mathbb{C}^2, V_{f'}) \) can be obtained by a sequence of quadratic transformations. Replacing the quadratic transformations of the infinitely near points of \( 0 \in \mathbb{C}^2 \) by blow ups along the infinitely near 1-dimensional axis of the \( z \)-axis, one obtains the following picture.

Let \( \Gamma_{\mathbb{C}^2; f} \) denote the minimal embedded resolution graph of the plane curve singularity \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \). Recall that, besides the Euler numbers and genera of the non-arrowheads, each vertex has a multiplicity decoration \( (m) \), the vanishing order of the pull-back of \( f' \) along the corresponding irreducible curve.

We say that \( \{ f = 0 \} \) is the cylinder of the plane curve \( \{ f' = 0 \} \).

In this situation, one can get a possible dual graph \( \Gamma_{\mathbb{C}^2} \) from \( \Gamma_{\mathbb{C}^2; f'} \) via the following conversion.

The shapes of the two graphs agree, only the decorations are modified: the Euler numbers are deleted, while for each vertex the multiplicity \( (m) \) is replaced by \( (m; 0, 1) \). The genus decorations in \( \Gamma_{\mathbb{C}^2} \) – similarly as in \( \Gamma_{\mathbb{C}^2; f'} \) – of all non-arrowheads are zero. Moreover, all edges in \( \Gamma_{\mathbb{C}^2} \) have weight 2.

**Example 9.1.1.** Let \( f(x, y, z) = f'(x, y) = (x^2 - y^3)(y^2 - x^3) \). Then \( \Gamma_{\mathbb{C}^2; f'} \) is:

```
(5) -- (-2) -- (1)
  |         |         |
  |         |         |
(10) -- (-1) -- (4)
  |         |         |
(10) -- (5) -- (-2)
```

(10)
which is transformed into $\Gamma_\mathcal{E}$ as:

\[\begin{array}{cccccc}
(5; 0, 1) & (10; 0, 1) & (4; 0, 1) & (10; 0, 1) & (5; 0, 1) \\
2 & 2 & 2 & 2 & 2 \\
(1; 0, 1) & (1; 0, 1)
\end{array}\]

9.1.2. It is easy to verify that $\Gamma_\mathcal{E}^2 = \Gamma_\mathcal{E}$, and $\Gamma_\mathcal{E}^1$ consists of $|\mathcal{A}|$ double arrows, where $|\mathcal{A}|$ is the number of irreducible components of $f'$. The statements of 7.3 and 7.4 can easily be verified.

9.2 Germs of Type $f = zf'(x, y)$

Here $f' : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is an isolated plane curve singularity as above, $f(x, y, z) := zf'(x, y)$ and $g$ is a generic linear form in variables $(x, y, z)$.

For this family we found no nice uniform presentation of $\Gamma_\mathcal{E}$ with similar simplicity and conceptual conciseness as in Sect. 9.1, or in the homogeneous case. (We face the same obstruction as in the case of suspensions, explained in the second paragraph of 9.3.1). Since the 3-manifold $\partial F$ can be determined completely and rather easily for any $f = zf'(x, y)$ by another method, which will be presented in Chap. 21, we decided to omit general technical graph-presentations here. Nevertheless, particular testing examples can be determined without difficulty. For example, consider $f' = x^{d-1} + y^{d-1}$ when $f$ becomes homogeneous and $\Gamma_\mathcal{E}$ can be determined as in Chap. 8. Or, consider $f' = x^2 + y^3$, whose $\Gamma_\mathcal{E}$ is below. For more comments (and mysteries) regarding the possible graphs $\Gamma_\mathcal{E}$, see 21.1.8.

Example 9.2.1. Assume that $f = z(x^2 + y^3)$ and take $g$ to be a generic linear form. The “ad hoc blowing up procedure”, using the naive principle to blow up the “worst singular locus”, provides the following $\Gamma_\mathcal{E}$, where we only marked the 2-edges, and all unmarked edges are 1-edges:

\[\begin{array}{cccc}
(1; 4, 1) & (1; 8, 2) & (1; 3, 1) & (1; 0, 1) \\
2 & 2 & 2 & 2 \\
(1; 7, 1) & (1; 0, 1) \\
(3; 7, 1) & (6; 7, 1) & (6; 3, 1) & (2; 3, 1)
\end{array}\]