Chapter 4
Phenomenology of ADMs

“The ultimate goal, however, must be a rational theory of statistical hydrodynamics where....properties of turbulent flows can be mathematically deduced from the fundamental equations of hydromechanics.” - E. Hopf

4.1 Basic Properties of ADMs

An approximate deconvolution operator denoted by $D$ is an approximate filter inverse that is accurate on the smooth velocity components and does not magnify the rough components.

Definition 37. The higher order generalized fluctuation is

$$w^* := w - D(w)$$

Given a chosen deconvolution operator $D$, the associated ADMs are given by

$$w_t + \nabla \cdot (D_N w D_N w) - \nu \Delta w + \nabla q + \chi w^* = f, \text{ and } \nabla \cdot w = 0. \quad (4.1)$$

The $w^*$ term is included to damp strongly the temporal growth of the fluctuating component of $w$ driven by noise, numerical errors, and inexact boundary conditions. The two simplest examples of such a models arise:

(a) when $N = 0$ and the $w^*$ term is dropped. This zeroth order model also arises as the zeroth order model in many different families of LES models. It is given by

$$w_t + \nabla \cdot (w w) - \nu \Delta w + \nabla q = f, \text{ and } \nabla \cdot w = 0. \quad (4.2)$$

(b) when no closure model is used on the nonlinear term and the time relaxation regularization is added
\[ w_t + \nabla \cdot (w w) - \nu \Delta w + \nabla q + \chi w^* = f, \text{ and } \nabla \cdot w = 0. \] (4.3)

The Stolz-Adams-Kleiser ADMs are highly accurate, in the sense that their consistency error is asymptotically small as \( \delta \) approaches zero, they are reversible, conserve model energy and helicity, and can be shown to be approximately Galilean invariant to high order. Their usefulness thus hinges on their stability properties.

To see the mathematical key to the estimates of energy and helicity dissipation rates we first recall from the energy equality for the ADM \([DE06]\). Paralleling the case of the zeroth order model, the natural idea is to make the nonlinear term disappear in the energy inequality by multiplying the model by \( AD_N w \) then integrating over the flow domain and integrating by parts. To give a hint of why this procedure is hopeful, we note that, in effect, this is simply re-norming the space \( L^2(\Omega) \).

Recall the notation for the deconvolution weighted norm and inner product,

\[ (\phi, \psi)_N := (\phi, D_N \psi), \quad \| \phi \|_N := (\phi, \phi)_N^{1/2} = (\phi, D_N \phi)_N^{1/2}. \]

**Lemma 38.** The norm \( \| v \|_{D_N} \) is a norm on \( L^2(\Omega) \) which is equivalent to the \( L^2(\Omega) \) norm.

**Proof.** This follows from the fact that the transfer function of \( D_N \), \( \hat{D}_N \), is bounded and bounded away from zero. \( \square \)

**Proposition 39.** If \( w \) is a weak or strong solution\(^1\), \( w \) satisfies

\[
\frac{1}{2} \left[ \| w(T) \|_{D_N}^2 + \delta^2 \| \nabla w(T) \|_{D_N}^2 \right] + \int_0^T \nu \| \nabla w(t) \|_{D_N}^2 + \nu \delta^2 \| \Delta w(t) \|_{D_N}^2 dt = \frac{1}{2} \left[ \| w_0 \|_{D_N}^2 + \delta^2 \| \nabla w_0 \|_{D_N}^2 \right] + \int_0^T (f, w(t))_N dt.
\]

**Proof.** (Sketch) Let \( (w, q) \) denote a periodic solution of the ADM (1.4). Multiplying by \( AD_N w \) and integrating over \( \Omega \) gives

\(^1\)Unlike the NSE case, it is known that weak=strong for the ADM and both exist and are unique.