In this and the following chapter, we present the conservation laws of fluid mechanics that are necessary to understand the basics of flow physics in turbomachinery from a unified point of view. The main subject of this chapter is the differential treatment of the conservation laws of fluid mechanics, namely conservation law of mass, linear momentum, angular momentum, and energy. These subjects are treated comprehensively in a recent book by Schobeiri [1]. In many engineering applications, such as in turbomachinery, the fluid particles change the frame of reference from a stationary frame followed by a rotating one. The absolute frame of reference is rigidly connected with the stationary parts, such as casings, inlets, and exits of a turbine, a compressor, a stationary gas turbine or a jet engine, whereas the relative frame is attached to the rotating shaft, thereby turning with certain angular velocity about the machine axis. By changing the frame of reference from an absolute frame to a relative one, certain flow quantities remain unchanged, such as normal stress tensor, shear stress tensor, and deformation tensor. These quantities are indifferent with regard to a change of frame of reference. However, there are other quantities that undergo changes when moving from a stationary frame to a rotating one. Velocity, acceleration, and rotation tensor are a few. We first apply these laws to the stationary or absolute frame of reference, then to the rotating one.

The differential analysis is of primary significance to all engineering applications such as compressor, turbine, combustion chamber, inlet, and exit diffuser, where a detailed knowledge of flow quantities, such as velocity, pressure, temperature, entropy, and force distributions, are required. A complete set of independent conservation laws exhibits a system of partial differential equations that describes the motion of a fluid particle. Once this differential equation system is defined, its solution delivers the detailed information about the flow quantities within the computational domain with given initial and boundary conditions.

### 3.1 Mass Flow Balance in Stationary Frame of Reference

The conservation law of mass requires that the mass contained in a material volume \( \mathbf{v} = \mathbf{v}(\mathbf{r}) \) must be constant:

\[
\dot{m} = \int_{\mathbf{v}_0} \rho d\mathbf{v}
\]  \hspace{1cm} (3.1)

Consequently, Eq. (3.1) requires that the substantial changes of the above mass must disappear:
Using the Reynolds transport theorem (see Chapter 2), the conservation of mass, Eq. (3.2), results in:

\[
\frac{D}{Dt} \int_{\Omega(t)} \rho \, dv = 0
\]  

(3.2)

Since the integral in Eq. (3.3) is zero, the integrand in the bracket must vanish identically. As a result, we may write the continuity equation for unsteady and compressible flow as:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0
\]  

(3.4)

Equation (3.4) is a coordinate invariant equation. Its index notation in the Cartesian coordinate system given is:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho V_i)}{\partial x_i} = 0
\]  

(3.5)

Expanding Eq. (3.5), we get:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho V_1)}{\partial x_1} + \frac{\partial (\rho V_2)}{\partial x_2} + \frac{\partial (\rho V_3)}{\partial x_3} = 0
\]  

(3.6)

For an orthogonal curvilinear coordinate system, the continuity equation (3.6) is written as (see Appendix A.34a):

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = \frac{\partial \rho}{\partial t} + (\rho V^i)_{,i} + (\rho V^j)\Gamma^i_{ij} = 0
\]  

(3.7)

Applying Eq. (3.7) to a cylindrical coordinate system with the Christoffel symbols, Eq. (3.8), from Appendix (B.61)

\[
\begin{align*}
\Gamma^1_{lm} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \quad \Gamma^2_{lm} &= \begin{pmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \quad \Gamma^3_{lm} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]  

(3.8)