Exploring the Foundations of Discrete Analytical Geometry in Isabelle/HOL

Jacques Fleuriot

Centre for Intelligent Systems and their Applications
School of Informatics, University of Edinburgh,
EH8 9AB, United Kingdom
jdf@inf.ed.ac.uk

Abstract. This paper gives an overview of the formalization of the Harthong-Reeb integer number system ($\text{HR}_\omega$) in the proof-assistant Isabelle. The work builds on an existing mechanization of nonstandard analysis and describes how the basic notions underlying $\text{HR}_\omega$ can be recovered and shown to have their expected properties, without the need to introduce any axioms. We also look at the formalization of the well-known Euler method over the new integers and formally prove that the algorithmic approximation produced can be made to be infinitely-close to its continuous counterpart. This enables the discretization of continuous functions and of geometric concepts such as the straight line and ellipse and acts as the starting point for the field of discrete analytical geometry.

Keywords: discrete geometry, nonstandard analysis, Harthong-Reeb numbers, Euler method, mechanical theorem proving, Isabelle.

1 Introduction

The Harthong-Reeb number system $\text{HR}_\omega$ is a well-established discrete model of the continuum [10] stemming from nonstandard analysis (NSA) [13]. It can provide a systematic framework in which a continuous function, e.g. an ellipse, can be discretized and the results then shown to be equivalent to the original continuous function through the use of rigorous arguments based on infinitesimals and infinitely large numbers.

In this work, we carry out a rigorous formalization of $\text{HR}_\omega$ in the proof assistant Isabelle [13]. This is both an exercise in formalized mathematics and a means of extending the number systems available for reasoning about geometric problems in the theorem prover. In some past work [7,4], we already combined NSA with geometry to formally explore the properties of novel infinitesimal and infinite geometric notions. Although the current work is in the same spirit, we do not claim to have established any of the mathematical foundations but merely to be verifying some of the interesting results that a team of people have been producing recently in discrete geometry [12,9].

As a result, our article deliberately follows in structure much of the exposition given by Chollet et al. [2], which purposefully revisits the original NSA approach.
to discrete analytical geometry (DAG). We examine their approach and show that their results can be rigorously mechanized in Isabelle/HOL as a conservative extension of our existing theories i.e. without introducing any new axioms.

2 On Nonstandard Analysis in Isabelle

The development of nonstandard analysis in Isabelle [8] is based on the extensional approach, first introduced by Robinson [15]. The system contains new types of numbers which are nonstandard extensions of the usual number systems. Thus, the hypernaturals $\mathbb{N}^*$ and hyperreals $\mathbb{R}^*$ are the extensions of the natural $\mathbb{N}$ and real numbers $\mathbb{R}$, respectively, and contain new, well-defined notions such as infinitely large numbers and infinitesimals. In Isabelle, all the nonstandard number systems are obtained through the so-called ultrapower construction [8].

While we shall not delve into the details of the development of NSA in Isabelle (the interested reader may consult a number of papers on this [8,6]), we wish to note that this approach differs from the one that is usually used in the presentation of the Harthong-Reeb numbers, where a minimal, axiomatic form of nonstandard analysis, related to Nelson’s Internal Set Theory (IST) [12], is preferred [1]. We note also that the aim of the current work is not to advocate our version of NSA as an alternative to the one that is usually used but to show that the Harthong-Reeb number system, its properties, and use can be formalized as conservative extensions of our existing mechanization.

2.1 Nonstandard Numbers

In the axiomatic version of NSA, a new predicate $\text{lim}(x)$ is introduced to indicate that a number $x$ is limited (the predicate $\text{standard}(x)$ is also often used e.g. in IST). Informally, this enables the theory to distinguish between what the extensional version of NSA would classify as finite and infinite nonstandard numbers. If we consider the hypernaturals, for instance, then the limited numbers are just the familiar natural numbers — denoted by $\text{Nats}$ in Isabelle — while the non-limited numbers correspond to the infinitely large hypernaturals, denoted by $\text{HNatInfinite}$ in Isabelle. Using this idea, we can straightforwardly define the set of limited hyperreal numbers in Isabelle/HOL:

$$\text{Limited} = \{ x :: \text{hypreal}. \exists n \in \text{Nats}. |x| < \text{hypreal_of_hypnat } n \}$$

where $\text{hypreal_of_hypnat } (n)$ is the function that embeds a limited hypernatural number (cf. a finite natural number) in the hyperreals. We note here the need for such embedding functions that enable one type of number to be mapped into another type. These are pervasive to our formalization as the simply-typed system of Isabelle does not allow subtyping. For the rest of this paper, though, unless essential, we shall omit these functions from our descriptions.

1 In Isabelle, $x :: \tau$ means that $x$ is of type $\tau$. 