Chapter 6
Some Methods Based on the HAM

Abstract In this chapter, some analytic and semi-analytic techniques based on the homotopy analysis method (HAM) are briefly described, including the so-called “homotopy perturbation method”, the optimal homotopy asymptotic method, the spectral homotopy analysis method, the generalized boundary element method, and the generalized scaled boundary finite element method. The relationships between these methods with the HAM are also revealed.

6.1 A brief history of the homotopy analysis method

To reveal the relationship between the homotopy analysis method (HAM) (Liao, 1992, 1997a, 1999, 2003, 2004; Liao and Tan, 2007; Liao, 2010) with other analytic approximation methods, we first briefly describe the basic ideas of the HAM and its history of development and modification.

The early HAM was first described by Shijun Liao (1992) in his PhD dissertation. For a given nonlinear differential equation

$$\mathcal{N}[u(x)] = 0, \quad x \in \Omega,$$

where $\mathcal{N}$ is a nonlinear operator and $u(x)$ is an unknown function, Liao (1992) used the concept of homotopy (Hilton, 1953) in topology (Sen, 1983) to construct a one-parameter family of equations in the embedding parameter $q \in [0, 1]$, called the zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(x;q) - u_0(x)] + q \mathcal{N}[\phi(x;q)] = 0, \quad x \in \Omega, \quad q \in [0, 1], \quad (6.1)$$

where $\mathcal{L}$ is an auxiliary linear operator and $u_0(x)$ is an initial guess. In theory, the concept of homotopy (Hilton, 1953) in topology (Sen, 1983) provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$. At $q = 0$ and $q = 1$, we have $\phi(x;0) = u_0(x)$ and $\phi(x;1) = u(x)$, respectively. So, as the embedding parameter $q \in [0, 1]$ increases from 0 to 1, the solution $\phi(x;q)$ of the
zeroth-order deformation equation (6.1) varies (or deforms) from the initial guess $u_0(x)$ to the exact solution $u(x)$ of the original nonlinear equation $\mathcal{N}[u(x)] = 0$. Such kind of continuous variation is called deformation in topology, so that (6.1) is called the zeroth-order deformation equation. Since $\phi(x;q)$ is also dependent upon the embedding parameter $q \in [0,1]$, we can expand it into the Maclaurin series with respect to $q$:

$$
\phi(x;q) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x) q^n,
$$

(6.2)
called the homotopy-Maclaurin series. Assuming that, the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$ are so properly chosen that the above homotopy-Maclaurin series converges at $q = 1$, we have the so-called homotopy-series solution

$$
u(x) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x),
$$

(6.3)
which satisfies the original equation $\mathcal{N}[u(x)] = 0$, as proved by Liao (1999, 2003) in general.

The governing equation of $u_n(x)$ is completely determined by the zeroth-order deformation equation (6.1). Differentiating (6.1) $n$ times with respect to the embedding parameter $q$, then dividing by $n!$ and finally setting $q = 0$, we have the so-called high-order deformation equation

$$
\mathcal{L} \left[ u_n(x) - \chi_n u_{n-1}(x) \right] = -\mathcal{D}_{n-1} \{ \mathcal{N}[\phi(x;q)] \},
$$

(6.4)
where $\chi_1 = 0$ and $\chi_k = 1$ when $k \geq 2$, $\mathcal{D}_k$ is the so-called $k$th-order homotopy-derivative operator defined by

$$
\mathcal{D}_k = \frac{1}{k!} \frac{\partial^k}{\partial q^k} \bigg|_{q=0}.
$$

(6.5)
Note that the high-order deformation equation (6.4) is always linear with the known term on the right-hand side, therefore is easy to solve, as long as we choose the auxiliary linear operator $\mathcal{L}$ properly.

Unfortunately, the early HAM mentioned above can not guarantee the convergence of approximation series of nonlinear equations in general. To overcome this restriction, Liao (1997a) generalized the concept of homotopy and introduced such a non-zero auxiliary parameter $c_0$ to construct a two-parameter family of equations, i.e. the zeroth-order deformation equation

$$
(1 - q) \mathcal{L} \left[ \phi(x;q) - u_0(x) \right] = c_0 \ q \ \mathcal{N}[\phi(x;q)], \ \ x \in \Omega, \ q \in [0,1].
$$

(6.6)
In this way, the homotopy-series solution (6.3) is dependent upon not only the physical variable $x$ but also the auxiliary parameter $c_0$. It has been proved (for details, please refer to Chapter 5) that the auxiliary parameter $c_0$ can adjust and control the convergence region of homotopy-series solutions, although $c_0$ has no physi-