ON THE SIZE OF A STABLE MINIMAL SURFACE IN $R^3$.

By J. L. Barbosa* and M. do Carmo.*

1. Notations and results.

1.1. Let $M$ be a two-dimensional, orientable $C^\infty$-manifold. A domain $D \subset M$ is an open, connected subset with compact closure $\overline{D}$ and such that the boundary $\partial D$ is a finite union of piece-wise smooth curves. Let $x: M \rightarrow R^3$ be a minimal immersion into the Euclidean space $R^3$. It is well known that $D$ is a critical point of the area of the induced metric, for all variations of $\overline{D}$ which keep $\partial D$ fixed. When this critical point is a minimum for all such variations, we say that $D$ is stable. The goal of this paper is to estimate the “size” of a stable minimal immersion and the main theorem is as follows. Set $S^2_1 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ and denote by $g: M \rightarrow S^2_1$, the Gauss map of the immersion $x$.

**Theorem 1.2.** Let the area of the spherical image $g(D) \subset S^2_1$ of a domain $D \subset M$ be smaller than $2\pi$. Then $D$ is stable.

This estimate is sharp, as can be shown, for instance, by considering pieces of the catenoid bounded by circles $C_1$ and $C_2$ parallel to and in opposite sides of the waist circle $C_0$. By choosing $C_1$ close to $C_0$ and $C_2$ far from $C_0$, we may obtain examples of unstable domains whose spherical image has area larger than $2\pi$ and as close to $2\pi$ as we wish. Further details will be given in Section 2.

Since $g$ may cover $g(D)$ more than once, Theorem 1.2 implies (but it is stronger than) that if the total curvature is smaller than $2\pi$, then $D$ is stable.

Let $N$ be a unit normal field along $x(M)$. Let $\Delta =$ divgrad and $K$ denote the Laplacian and the Gaussian curvature of $M$, respectively, in the induced metric. Given a piece-wise smooth function $u: \overline{D} \rightarrow R$, with $u \equiv 0$ on $\partial D$, the second derivative of the area function for a variation whose deformation vector field is given by $V = uN$ is (Cf. 3.2.3 of [9]).

\[ I(V, V) = \int_D u(-\Delta u + 2uK) dM. \]  

(1.3)

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where $dM$ is the element of area of $M$ in the induced metric. If $I(V, V) > 0$, for all such $V$, then $D$ is stable. We say that $D$ is unstable if for some $V, I(V, V) < 0$.

A Jacobi field in $\overline{D}$ is a normal field $uN$, where $u : \overline{D} \to \mathbb{R}$ is a smooth function which satisfies

$$-\Delta u + 2uK = 0. \quad (1.4)$$

A boundary $\partial D$ of a domain $D \subset M$ is a conjugate boundary if there exists a non-zero Jacobi field on $\overline{D}$ vanishing on $\partial D$; if, in addition, there exists no domain $D' \subset D, D' \neq D$, such that $\partial D'$ is a conjugate boundary, $\partial D$ is called a first conjugate boundary. The multiplicity of a conjugate boundary $\partial D$ is the number of linearly independent Jacobi fields on $\overline{D}$ vanishing on $\partial D$.

Theorem 1.2 is related to some results of A. H. Schwarz (see [8]). In Section 2 we prove these results in our context. We also give a simple proof of the fact that the multiplicity of a first conjugate boundary is one.

In Section 3 we prove Theorem 1.2 and indicate another application of the ideas of the proof. The section closes with a few open questions.

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2. A result of A. H. Schwarz

2.1. Let $W$ be a two-dimensional real analytic Riemannian manifold and let $D \subset W$ be a domain in $W$. We will denote by $\Delta_w$ the Laplacian of $W$ and by $H(D)$ the space of $C^\infty$ functions on $\overline{D}$ which are not identically zero and vanish on $\partial D$. A real number $\lambda > 0$ such that there exists a solution of $\Delta_w u + \lambda u = 0, u \in H(D)$, is called an eigenvalue in $D$ for $\Delta_w$ (this is actually the negative of the usual eigenvalue). The space

$$P_\lambda(D) = \{ u \in H(D); \Delta_w u + \lambda u = 0 \}$$

of such solutions is the eigenspace corresponding to $\lambda$. It is known that if $u \in P_\lambda(D), u$ is analytic in $D$. It is also known that the eigenvalues in $D$ form a discrete set of positive numbers, and, as usual, we order then so that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \ldots .$$

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