COMPACT CONFORMALLY FLAT HYPERSURFACES

BY

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ABSTRACT. Roughly speaking, a conformal space is a differentiable manifold $M^n$ in which the notion of angle of tangent vectors at a point $p \in M^n$ makes sense and varies differentiably with $p$; two such spaces are (locally) equivalent if they are related by an angle-preserving (local) diffeomorphism. A conformally flat space is a conformal space locally equivalent to the euclidean space $R^n$. A submanifold of a conformally flat space is said to be conformally flat if so its induced conformal structure; in particular, if the codimension is one, it is called a conformally flat hypersurface.

The aim of this paper is to give a description of compact conformally flat hypersurfaces of a conformally flat space. For simplicity, assume the ambient space to be $R^{n+1}$. Then, if $n \geq 4$, a conformally flat hypersurface $M^n \subset R^{n+1}$ can be described as follows. Diffeomorphically, $M^n$ is a sphere $S^n$ with $b_1(M)$ handles attached, where $b_1(M)$ is the first Betti number of $M$. Geometrically, it is made up by (perhaps infinitely many) nonumbilic submanifolds of $R^{n+1}$ that are foliated by complete round $(n-1)$-spheres and are joined through their boundaries to the following three types of umbilic submanifolds of $R^{n+1}$: (a) an open piece of an $n$-sphere or an $n$-plane bounded by round $(n-1)$-sphere, (b) a round $(n-1)$-sphere, (c) a point.

1. Introduction.

1.1. The aim of this paper is to present a rather complete description of compact conformally flat hypersurfaces of a simply-connected, $(n + 1)$-dimensional space form, for $n \geq 4$. Before stating our results, we will recall some known facts and definitions, mainly in order to fix our notation.

Manifolds are $C^\infty$ and boundaryless. A Riemannian manifold $M^n$ is (locally) conformally flat if for each point $p \in M^n$ there exists an open neighborhood $V$ of $p$ in $M$ and a conformal diffeomorphism of $V$ onto an open set of $R^n$ (superscripts will denote dimensions and will be dropped when clear from the context). An immersion $x\colon M^n \to \bar{M}^{n+k}$ of a differentiable manifold into a Riemannian manifold $\bar{M}^{n+k}$ is a conformally flat immersion if $M^n$ is conformally flat with respect to the induced metric; if $k = 1$, we will say that $x$ is a conformally flat hypersurface.

1.2. The following notation will be used throughout the paper.

Let $M^n$ be a differentiable manifold and $x\colon M^n \to \bar{M}^{n+k}$ an immersion into a Riemannian manifold $\bar{M}^{n+k}$. We will denote by $\bar{\nabla}$ the Riemannian connection on $\bar{M}$ and by $\nabla$ the induced connection on $M$. Let $X$ and $Y$ be local tangent vector...
fields along \( x \); then
\[
\alpha(X, Y) = \nabla_X Y - \nabla_Y X
\]
will denote the second fundamental form of \( x \). By fixing a point \( p \in M \) and a local unit normal vector field \( N \) along \( x \) in a neighborhood of \( p \), \( \alpha \) determines a selfadjoint linear map \( A_N: T_p M \to T_p M \) of the tangent space \( T_p M \) of \( M \) at \( p \), namely,
\[
\langle A_N(X), Y \rangle = \langle \alpha(X, Y), N \rangle.
\]
We will call \( A_N \) the Weingarten operator of \( x \) relative to \( N \). When \( k = 1 \), we will call \( N \) a local orientation at \( p \). Given a local orientation \( N \), we will denote by \( \lambda_1, \ldots, \lambda_n \) the eigenvalues of \( A_N \). We will use \( M(c) \) to denote a Riemannian manifold with constant curvature \( c \); when such a manifold is, in addition, complete and simply-connected, it will be denoted by \( \tilde{M}(c) \).

For \( n \geq 4 \) it is well known that \( x: M^n \to M^{n+1}(c) \) is conformally flat if and only if, after a possible re-enumeration, \( \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} \) (see Cartan [1] or, for a simple proof, see e.g. [7]); notice that this condition does not depend on the choice of a local orientation. For convenience, we will set \( \lambda_1 = \lambda, \lambda_n = \mu \) and will denote by \( U = \{ p \in M; \lambda = \mu \} \) the set of umbilic points of \( M \). We will denote by \((a, b)\) an open interval of the real line.

1.3. In \( \S 2 \) we will prove our first main result which describes the geometric structure of a compact conformally flat hypersurface \( x: M^n \to \tilde{M}^{n+1}(c), n \geq 4 \). To simplify the statement, let us call a compact submanifold \( \Sigma^k \subset M^n \) a \( k \)-sphere if the restriction \( x|\Sigma^k: \Sigma^k \to \tilde{M}^{n+1}(c) \) is a totally umbilic immersion.

1.4 THEOREM. Let \( x: M^n \to \tilde{M}^{n+1}(c) \) be an immersed conformally flat hypersurface, \( n \geq 4 \), \( M^n \) compact and connected. Assume that \( U \neq \emptyset \). Then:

(i) Each connected component \( D \) of the set of umbilic points admits a codimension one foliation by \((n-1)\)-spheres; in particular \( D \) is diffeomorphic to \( S^{n-1} \times (a, b) \). Furthermore, the boundary of \( D \) has at most two connected components and each such component is either a point, an \((n-1)\)-sphere or two \((n-1)\)-spheres with a common point.

(ii) Each connected component of \( U \) is a point, an \((n-1)\)-sphere, or an \( n \)-dimensional umbilic set bounded by points or by a union of \((n-1)\)-spheres such that two such spheres have at most a common point.

A more precise description of the possible shapes of the boundary of \( D \) is as follows (see Figure 1): It consists of two points (the closure \( \overline{D} \) of \( D \) is then diffeomorphic to an \( n \)-sphere, Figure 1(a)), or one round \((n-1)\)-sphere (\( \overline{D} \) is diffeomorphic to \( S^{n-1} \times S^1 \)—Figure 1(b)—or to a generalized Klein bottle), or one point and one round \((n-1)\)-sphere (\( \overline{D} \) is diffeomorphic to an \( n \)-ball, Figure 1(d)), or two round \((n-1)\)-spheres that either are disjoint (Figure 1(e)) or have one common point (Figure 1(c)). Thus we can think of a compact conformally flat hypersurface as nonumbilic “conformally flat handles” that are foliated by \((n-1)\)-spheres and are used to join points, \((n-1)\)-spheres, and pieces of umbilic \( n \)-submanifolds of \( \tilde{M}(c) \). A typical picture is given in Figure 2.