Chapter 5
Normal Distribution Binomial Heritage


5.1 Normal Statistics, Preliminaries

To understand how specific and how universal the normal distribution is, the point based on the Central Limit Theorems of the Theory of Probability should be taken. A routine course presenting this theory usually closes with the Central Limit Theorems. Therefore, the foregoing presentation may only present these results without supplying the Student with any rigorous proofs and leaving out the details. Thus, when indicating possible courses which present a similar approach, let us first mention a book by Weinberg [1] referred to earlier rather than that by Neyman [2], however, this remark is addressed more to the instructor than to the student. From an intuitional point of view a very important element of the limit theorems seems to be the fact that the normal distribution is the result of the sum of a number of random components (strictly speaking they are random variables) not necessarily of precisely defined nature (which stands for the knowledge of their distributions)\(^1\). How universal the normal distribution is has constituted a heated subject of discussions or even bitter quarrels among mathematicians and statisticians for more than a century. There is no doubt about its power, but there is also no doubt about its limitations. The human species displays a wide range of such applications, from the purely physical (stature or weight) to mental (such as IQ or grades). One of its special applications is mass products. From a theoretical point of view the normal distribution has a unique property: invariance regarding linear transformations. The first encounter with such a property was offered by Chapter 1.

\(^1\) Just here we may recall a rule of the thumb well known in Statistics and using at least twelve uniformly distributed components to get a sample of the normal distribution.
Below we commence with presenting the normal distribution from scratch. It is given in general form by (5.1) showing a real function of the real variable with two parameters denoted by $\bar{x}$, $\sigma^2$ and these symbols suggest the special meaning of the parameters. Of course $\bar{x}$ denotes the basic mean, and $\sigma^2$ denotes the variance.

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}} \quad \text{here: } -\infty < x < \infty$$ \hspace{1cm} (5.1)

The above stated properties of the normal distribution will be proved (see p.5.4) – here we limit ourselves to presenting the defining steps. The basic mean is formally defined by:

$$\mu = \int_{-\infty}^{+\infty} x \cdot f(x) \, dx$$ \hspace{1cm} (5.2)

To avoid a clash of symbols in (5.2) symbol $\mu$ also appears with respect to the first moment. The proof will justify that substitution of (5.1) by (5.2) will give:

$$\mu = \bar{x}$$ \hspace{1cm} (5.3)

To make (5.2) more familiar, let us recall the formal rule to derive the basic mean on the ground of the grouped data

$$\mu = \frac{1}{N} \sum_{i=1}^{N} f_i \cdot x_i$$

assuming that $f_i$ is replaced by $f(x)$ interprets (5.2) as the formula suitable for the continuous distribution such as normal distribution. Similar reasoning can be applied with respect to the formal rule deriving the variance:

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - \bar{x})^2 \cdot f(x) \, dx$$ \hspace{1cm} (5.4)

To be illustrated graphically the distribution given in (5.1) should be slightly modified. This modification means that the mean value has been assumed as zero. The result of such a transformation leads to the centered distribution (5.5):

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$ \hspace{1cm} (5.5)

We shall comment the content of Fig. 5.1 depicting three well known bell-shaped curves. It gives the best opportunity to explain the role of the variance particularly regarding normal distribution due to the fact, that Fig.5.1 presents three diagrams preserving the real scales in both coordinates showing appropriate numerical values. Regarding these values we are also going to pay attention.