Stable Splitting of Bivariate Splines Spaces by Bernstein-Bézier Methods

Oleg Davydov and Abid Saeed

Department of Mathematics and Statistics, University of Strathclyde, Glasgow, United Kingdom
{oleg.davydov, abid.saeed}@strath.ac.uk
http://www.mathstat.strath.ac.uk/

Abstract. We develop stable splitting of the minimal determining sets for the spaces of bivariate $C^1$ splines on triangulations, including a modified Argyris space, Clough-Tocher, Powell-Sabin and quadrilateral macro-element spaces. This leads to the stable splitting of the corresponding bases as required in Böhm’s method for solving fully nonlinear elliptic PDEs on polygonal domains.

Keywords: Fully nonlinear PDE, Monge-Ampère equation, multivariate splines, Bernstein-Bézier techniques.

1 Introduction

Numerical solution of fully nonlinear elliptic partial differential equations is a topic of intensive research and great practical interest, see [2,4]. Since no weak form formulation is available for the equations of this type in general, the standard Galerkin finite element method cannot be applied directly.

Recently, Böhm [1,2] introduced a general approach that solves the Dirichlet problem for fully nonlinear elliptic equations numerically with the help of a sequence of linear elliptic equations used within an appropriate Newton scheme. These linear elliptic equations can be solved by the finite element method, but the discretisation has to be done by appropriate spaces of $C^1$ finite elements (splines) that admit a stable splitting into a subspace satisfying zero boundary conditions, and its complement. Such a stable splitting has been developed in [6] for a modified space of the Argyris finite element.

In this paper we systematically study the problem of stable splitting for the spaces of bivariate $C^1$ splines on triangulations of low degree using the Bernstein-Bézier methods. It turns out that stable splitting can be easily formulated as splitting of the minimal determining sets (MDS). We revisit the modified Argyris space studied in [6] by a different technique, and show that its modification is necessary at least if the convenient MDS splitting approach is used. We also show that Clough-Tocher, Powell-Sabin and quadrilateral macro-element spaces admit the stable splitting and therefore can also be used in the Böhm’s numerical method.
The paper is organised as follows. Section 2 is devoted to an outline of Böhmér’s method, whereas Section 3 introduces necessary definitions from the theory of Bernstein-Bézier methods [8], and defines the stable splitting of an MDS. In Section 4 we discuss the stable splitting for the Argyris space and its modification, and Section 5 is devoted to the $C^1$ macro-element spaces.

2 Böhmér’s Method for Fully Nonlinear Elliptic PDEs

2.1 Fully Nonlinear Elliptic Operators

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and let $G : H^\gamma(\Omega) \rightarrow L^2(\Omega)$, $\gamma \geq 2$, be a second order differential operator of the form

$$G(u) = \tilde{G}(\cdot, u, \nabla u, \nabla^2 u),$$

where $\tilde{G}$ is a real valued function defined on a domain $\tilde{\Omega} \times \Gamma$ such that

$$\tilde{\Omega} \subset \tilde{\Omega} \subset \mathbb{R}^n \quad \text{and} \quad \Gamma \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n},$$

and $\nabla u, \nabla^2 u$ denote the gradient and the Hessian of $u$, respectively. The points in $\tilde{\Omega} \times \Gamma$ are denoted by $w = (x, z, p, r)$, with $x \in \tilde{\Omega}, z \in \mathbb{R}$, $p = [p_i]_{i=1}^n \in \mathbb{R}^n$, $r = [r_{ij}]_{i,j=1}^n \in \mathbb{R}^{n \times n}$, to indicate the product structure of this set.

The operator $G$ is said to be elliptic in a subset $\tilde{\Gamma} \subset \tilde{\Omega} \times \Gamma$ if the matrix $[\partial^2 \tilde{G} / \partial r_{ij}(w)]_{i,j=1}^n$ is well defined and positive definite for all $w \in \tilde{\Gamma}$ [27]. If $\tilde{G}$ is a linear function of $(z, p, r)$ for each fixed $x$, then $G$ is a linear differential operator. Under suitable restrictions on $\tilde{G}$, classes of quasilinear and semilinear differential operators are obtained [2] p. 80], but in general $G$ may be fully nonlinear.

In the neighborhood of a fixed function $\hat{u} \in H^\gamma(\Omega)$ the linear elliptic operator $G'(\hat{u})$ is defined by

$$G'(\hat{u})u = \frac{\partial \tilde{G}}{\partial z}(\hat{w})u + \sum_{i=1}^n \frac{\partial \tilde{G}}{\partial p_i}(\hat{w})\partial^i u + \sum_{i,j=1}^n \frac{\partial \tilde{G}}{\partial r_{ij}}(\hat{w})\partial^i \partial^j u,$$

where $\hat{w} = (x, \hat{u}(x), \nabla \hat{u}(x), \nabla^2 \hat{u}(x))$ is a function of $x \in \Omega$, and $\partial^i$ denotes the partial derivative with respect to the $i$-th variable. If $G : H^\gamma(\Omega) \rightarrow L^2(\Omega)$ is Fréchet differentiable at $\hat{u}$, then $G'(\hat{u}) : H^\gamma(\Omega) \rightarrow L^2(\Omega)$ is its Fréchet derivative. If $G'(\hat{u})$ depends continuously on $\hat{u}$ with respect to the linear operator norm, then $G$ is said to be continuously differentiable at $\hat{u}$.

Many nonlinear elliptic operators and corresponding equations $G(u) = 0$ are important for applications, for example the Monge-Ampère equation for $\Omega \subset \mathbb{R}^2$, given by

$$G_{MA}(u) := \det(\nabla^2 u) - f(x) = 0, \quad f(x) > 0 \text{ for } x \in \Omega.$$

The operator $G_{MA}$ is fully nonlinear and $G_{MA}(u) \in L^2(\Omega)$ if $u$ belongs to the Sobolev space $H^{5/2}(\Omega)$ and $f \in L^2(\Omega)$. Moreover, $G_{MA} : H^\gamma(\Omega) \rightarrow L^2(\Omega)$ is continuously differentiable if $\gamma \geq 5/2$. 