Chapter 13
Back to Classes

When the class is sparse, the teacher is far away...

In this chapter we summarize the results on sparsity of classes with all their characterizations. The multiplicity of the equivalent characterizations that can be given for the nowhere dense–somewhere dense dichotomy is mainly a consequence of several related aspects:

- The relationships between the different type of resolutions, namely of minor resolution, topological resolution, and immersion resolution;
- The relationships between shallow minors, shallow topological minors, lexicographic products and shallow immersions;
- The polynomial dependence (and weak polynomial dependence) of key graph invariants, like $\nabla_r$, $\tilde{\nabla}_r$, $\chi_p$, col$_p$, wcol$_p$, etc.;
- The characterization of uniformly quasi-wide classes.

This will be elaborated in detail in this chapter.

13.1 Resolutions

Our main classification, the nowhere dense–somewhere dense dichotomy, is based on resolutions. We defined several types of resolutions: the minor resolution

$$c^\nabla = (c \nabla 0, c \nabla 1/2, c \nabla 1, \ldots)$$

the topological resolution

$$c^{\tilde{\nabla}} = (c^{\tilde{\nabla}} 0, c^{\tilde{\nabla}} 1/2, c^{\tilde{\nabla}} 1, \ldots),$$

and the immersion resolution

\[ C^\gamma = (C \overset{\gamma}{\uparrow} (1, 0), C \overset{\gamma}{\uparrow} (2, 1/2), C \overset{\gamma}{\uparrow} (3, 1), \ldots). \]

As expected, these resolutions may behave differently. For instance, consider the minor resolution \( C^\gamma \) of a class \( C \) and its limit \( C \uparrow \infty \) (i.e. its minor closure):

The first possibility is that the class \( C \uparrow \infty \) is strictly included in Graph:

This is the case when \( C \) is included in a proper minor closed class. The minor resolution may then be used to get a finer information about subclasses with smaller density. For instance, if Planar is the class of all planar graphs and \( C_1 \) is the subclass of Planar with graphs of girth at least \( g \), then

\[ C_1 \uparrow \frac{1}{2} \lfloor \log_2 (g/3) \rfloor = C_1 \uparrow \infty = \text{Planar}. \]

However, if \( C_2 \) is the subclass of Planar with graphs of maximum degree 3, then \( C_2 \uparrow t \) is the class of planar graphs with maximum degree \( 3.2^t \) and thus the class Planar is only reached at the limit.

The second possibility is that the class \( C \uparrow \infty \) is equal to Graph, although \( C \uparrow t \) is strictly included in Graph for each \( t \):

This is the case when \( C \) is nowhere dense but is not included in a proper minor closed class. Such a situation allows us to parametrize graphs by the