Chapter 14
Classes with Bounded Expansion – Examples

Bounded expansion classes are the focus of this chapter and one of the leitmotivs of the whole book. In this chapter, we shall give many examples of classes with bounded expansion. The examples which we cover are schematically depicted on Fig. 14.1. These classes cover most classes considered in structural graph theory and the relevant parts of logic and discrete geometry. This will be explained for several of these classes in a greater detail in this chapter.

In Sect. 14.1, we show that the notion of bounded expansion is compatible with Erdős-Rényi model of random graphs with constant average degree (that is, for random graphs of order $n$ with edge probability $d/n$). Then, we provide a number of examples of classes with bounded expansion that appear naturally in the context of graph drawing or graph coloring. In particular, we prove that each of the following classes have bounded expansion, even though they are not contained in a (proper) topologically-closed class:

- Graphs that can be drawn with a bounded number of crossings per edge (Sect. 14.2),
- Graphs with bounded queue-number (Sect. 14.4),
- Graphs with bounded stack-number (Sect. 14.5),
- Graphs with bounded non-repetitive chromatic number (Sect. 14.6).

We also prove that graphs with “linear” crossing number are contained in a topologically-closed class, and graphs with bounded crossing number are contained in a minor-closed class (Sect. 14.2). Many of these results were obtained in collaboration with David Wood, see [359].
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Fig. 14.1 Classes with bounded expansion. The results about classes with bounded crossings, bounded queue-number, bounded stack-number, and bounded non-repetitive chromatic number are proved in this chapter. Arrows represent class inclusion.

14.1 Random Graphs (Erdős-Rényi Model)

The $G(n, p)$ model of random graphs was introduced by Gilbert [217] and Erdős and Rényi [170]. It is the most common random graph model, see e.g. [76]. In this model, a graph with $n$ vertices is built, with each edge appearing independently with probability $p$. It is frequently considered that $p$ may be a function of $n$, hence the notation $G(n, p(n))$ (see Fig. 14.2).

Let us review some basic facts about $G(n, d/n)$ and $G(n, p(n))$. The order of the largest complete (topological) minor in $G(n, p/n)$ was studied intensively. It is known since the work of [318] that random graphs $G(n, p(n))$ with $p(n) - 1/n \ll n^{-4/3}$ are asymptotically almost surely planar, whereas those with $p(n) - 1/n \gg n^{-4/3}$ asymptotically almost surely contain unbounded clique minors. Recall that a property of random graphs holds \textit{asymptotically}