Chapter 16
Applications to Elliptic Partial Differential Equations

Abstract We consider elliptic partial differential equations in \( d \) variables and their discretisation in a product grid \( I = \times_{j=1}^{d} I_j \). The solution of the discrete system is a grid function, which can directly be viewed as a tensor in \( V = \bigotimes_{j=1}^{d} K I_j \). In Sect. 16.1 we compare the standard strategy of local refinement with the tensor approach involving regular grids. It turns out that the tensor approach can be more efficient. In Sect. 16.2 the solution of boundary value problems is discussed. A related problem is the eigenvalue problem discussed in Sect. 16.3. We concentrate ourselves to elliptic boundary value problems of second order. However, elliptic boundary value problems of higher order or parabolic problems lead to similar results.

16.1 General Discretisation Strategy

The discretisation of partial differential equations leads to matrices whose size grows with increasing accuracy requirement. In general, simple discretisation techniques (Galerkin method or finite difference methods) using uniform grids yield too large matrices. Instead, adaptive discretisation techniques are used. Their aim is to use as few unknowns as possible in order to ensure a certain accuracy.

The first type of methods is characterised by the relation \( \varepsilon = O(n^{\kappa/d}) \), where \( \varepsilon \) is the accuracy of the approximation (in some norm), \( n \) number of degrees of freedom, \( \kappa \) the consistency order, and \( d \) the spatial dimension. The corresponding methods are Galerkin discretisations with polynomial ansatz functions of fixed degree. The relation \( \varepsilon = O(n^{\kappa/d}) \) is not always reached.\(^1\) Only if the solution behaves uniformly regular in its domain of definition, also the uniform grid yields

\[
\varepsilon = O(n^{\kappa/d}). \tag{16.1}
\]

\(^1\) For three-dimensional problems, edge singularities require stretched tetraeders. However, usual adaptive refinement strategies try to ensure form regularity (cf. [82]).

In the standard case, however, the solution of partial differential equations has point singularities and—for 3D problems—edge singularities. This requires a concentration of grid points towards corners or edges.

A more efficient method is the $hp$ finite element method in the case of piecewise analytic solutions. Here, the ideal relation between the error $\varepsilon$ and the number $n$ of unknowns is $\varepsilon = O(\exp(-\beta n^\alpha))$ for suitable $\alpha, \beta > 0$.

So far, for fixed $\varepsilon$, the strategy is to minimise the problem size $n$. In principle, this requires that also the generation of the system matrix and its solution is $O(n)$. This requirement can be relaxed for the $hp$ finite element method. If $n = O(\log^{1/\alpha} \varepsilon)$, even Gauss elimination with cost $O(\log^{3/\alpha} \varepsilon)$ is only a redefinition of $\alpha$ by $\alpha/3$.

Tensor applications require a Cartesian grid $I := I_1 \times \ldots \times I_d$ of unknowns. This does not mean that the underlying domain $\Omega \subset \mathbb{R}^d$ must be of product form. It is sufficient that $\Omega$ is the image of a domain $\Omega_1 \times \ldots \times \Omega_d$. For instance, $\Omega$ may be a circle, which is the image of the polar coordinates varying in $\Omega_1 \times \Omega_2$.

The use of a (uniform$^2$) grid $I = I_1 \times \ldots \times I_d$ seems to contradict the strategies from above. However, again the leading concept is: best accuracy for minimal cost, where the accuracy$^3$ is fixed by the grid $I$. For simplicity, we assume $n_j = \#I_j = n = O(\varepsilon^{-\beta})$. The storage cost of the tensor formats is $O(r^* nd)$, where $r^*$ indicates possible powers of some rank parameters. The approximation of the inverse by the technique from §9.7.2.6 costs $O(\log^2(1/\varepsilon) \cdot r^* nd)$. Comparing the storage and arithmetical cost with the accuracy, we see a relation like in (16.1), but the exponent $\kappa/d$ is replaced by some $\beta > 0$ independent of $d$.

A second step is the tensorisation from §14. As described in §14.2.3, the complexity reached by tensorisation corresponds (at least) to the $hp$ finite element approach. Therefore, in the end, the cost should be not worse than the best $hp$ method, but independent of $d$.

## 16.2 Solution of Elliptic Boundary Value Problems

We consider a linear boundary value problem

\[ Lu = f \text{ in } \Omega = \Omega_1 \times \ldots \times \Omega_d \subset \mathbb{R}^d, \quad u = 0 \text{ on } \partial \Omega \quad (16.2) \]

with a linear differential operator of elliptic type (cf. [82, §5.1.1]). Concerning the product form see the discussion from above. The homogeneous Dirichlet condition $u = 0$ on $\partial \Omega$ may be replaced by other conditions like the Neumann condition.

The standard dimension $d = 3$ is already of interest for tensor methods. The other extreme are dimensions of the order $d = 1000$.

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$^2$ The grids $I_j$ need not be uniform, but for simplicity this is assumed.

$^3$ In the case of a point singularity $r^\alpha (\alpha > 0, r = \|x - x_0\|, x_0: \text{corner point})$, the grid size $h$ leading to an accuracy $\varepsilon$ is of the form $h = O(\varepsilon^\beta), \beta > 0$. A uniform grid $I_j$ needs $n_j = O(\varepsilon^{-\beta})$ grid points.