We now consider idealized linear material behavior.

### 4.1 Linear Constitutive Equations

The starting point to develop a constitutive theory is to assume that an energy function per unit volume exists, a nonnegative function denoted $W$. A linear constitutive relation can be derived from

$$D = \frac{\partial W}{\partial E} \quad (4.1)$$

and

$$W \approx c_0 + c_1 \cdot E + \frac{1}{2} E \cdot \epsilon \cdot E + ... \quad (4.2)$$

which implies

$$D \approx c_1 + \epsilon \cdot E + ... \quad (4.3)$$

We are free to set $c_0 = 0$ (it is arbitrary) to have zero energy at zero $E$, and furthermore, we assume that no electric field fluxes exist in the reference state ($c_1 = 0$). Thus, we obtain a general anisotropic relation between the electric flux and the electric field:

$$D = \epsilon \cdot E. \quad (4.4)$$

This is a linear (tensorial) relation. A general material has nine independent constants since it is a second order tensor relating three components of electric flux to electric field. This is easily seen from the matrix representation of $\epsilon$
\[
\begin{bmatrix}
D_1 \\
D_2 \\
D_3
\end{bmatrix} = \begin{bmatrix}
\epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\
\epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\
\epsilon_{31} & \epsilon_{32} & \epsilon_{33}
\end{bmatrix} \begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}.
\]

(4.5)

The symbol \([\cdot]\) is used to indicate standard matrix notation equivalent to a tensor form. At this point, based on many factors that depend on the material microstructure, it can be shown that the components of \(\epsilon\) may be written in terms of anywhere between nine and one independent parameters. We explore such concepts further via the ideas of material symmetry.

**Remark:** The existence of a strictly positive stored energy function implies that the linear tensor must have positive eigenvalues at every point in the body. Typically, different materials are classified according to the number of independent constants in \(\epsilon\). The existence of a scalar energy function forces \(\epsilon\) to be symmetric, in other words,

\[
W = \frac{1}{2} E \cdot \epsilon \cdot E = \frac{1}{2} (E \cdot \epsilon \cdot E)^T = \frac{1}{2} E \cdot \epsilon^T \cdot E,
\]

which implies \(\epsilon^T = \epsilon\). Consequently, \(\epsilon\) has only six free constants. The non-negativity of \(W\) imposes the restriction that \(\epsilon\) remains positive definite.

### 4.1.1 Material Symmetry

Transformation matrices are used in determining the symmetries. Consider a plane of symmetry, the \(x_2 - x_3\) (Cartesian) plane (Figure 2.1). A plane of symmetry implies that the material has the same properties with respect to that plane. Therefore, we should be able to flip the axes with respect to that plane, and have no change in the constitutive law. By definition, a plane of symmetry exists at a point where the material constants have the same value for a pair of coordinate systems. The axes are referred to as "equivalent permittivity directions." Also by definition, an axis of symmetry of order \(K\) exists at a point when there are sets of equivalent permittivity directions which can be superposed by a rotation through an angle \(2\pi/K\) about an axis. The way to determine permittivity symmetry is as follows. First, we rotate the vectors

\[
[D] = [Q][D]
\]

(4.7)

and

\[
[E] = [Q][E],
\]

(4.8)

and thus the transformed system is

\[
[D] = [\epsilon][E],
\]

(4.9)