3
Differentiation Theory of the Functions

3.1 Partial Derivatives and Differentiable Functions of Several Variables

3.1.1 Partial Derivatives

Definition 3.1 (see [8]). A variable quantity $z$ is called a single-valued function of the two variables $x$, $y$, if there corresponds to each set of their values $(x, y)$ in a given range a unique value of $z$. The variables $x$ and $y$ are called arguments or independent variables. The functional relation is denoted by

$$z = f(x, y), \quad f : \mathbb{R}^2 \rightarrow \mathbb{R}. \quad (3.1)$$

We can define functions of three or more arguments in the same way.

Definition 3.2 (see [41], p. 166). If $z = f(x, y)$, then assuming, for example, that $y$ is constant, we get the derivative

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \lim_{x \to a} \frac{f(x, b) - f(a, b)}{x - a} = f'_x(x, y), \quad (3.2)$$

which is called the partial derivative of the function $z$ with respect to the variable $x$.

In a similar way, we define and denote the partial derivative of the function $z$ with respect to the variable $y$.
Definition 3.3 (see [15], p. 144). If they exist, the partial derivatives of the functions \( f'_x \) and \( f'_y \) they are known as second order partial derivatives and one denotes by:

\[
\begin{align*}
  f''_{xx} &= (f'_x)'_x = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} \\
  f''_{xy} &= (f'_x)'_y = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x \partial y} \\
  f''_{yx} &= (f'_y)'_x = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y \partial x} \\
  f''_{yy} &= (f'_y)'_y = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}.
\end{align*}
\] (3.3)

Proposition 3.4 (Schwarz’s criterion, see [15], p. 144). If the function \( f \) has the mixed second order partial derivatives \( f''_{xy} \) and \( f''_{yx} \) in a neighborhood of the point \( (a, b) \in A \subset \mathbb{R}^2 \) and if \( f''_{xy} \) and \( f''_{yx} \) are continuous in \( (a, b) \) then

\[ f''_{xy} (a, b) = f''_{yx} (a, b). \]

Example 3.5. Let \( f : \mathbb{R}^2 \to \mathbb{R} \), be specified as

\[ f(x, y) = \sqrt{\sin^2 x + \sin^2 y}. \]

Find the first partial derivatives of the function \( f \) at the points \( \left( \frac{\pi}{4}, 0 \right) \) and \( \left( \frac{\pi}{4}, \frac{\pi}{4} \right) \).

Solution.

We shall have

\[
\begin{align*}
  f'_x (x, y) &= \frac{\partial f}{\partial x} (x, y) = \frac{2 \sin x \cos x}{2 \sqrt{\sin^2 x + \sin^2 y}} = \frac{\sin x \cos x}{\sqrt{\sin^2 x + \sin^2 y}} \\
  f'_x \left( \frac{\pi}{4}, 0 \right) &= \frac{\sqrt{2}}{2} = 0.7071; \quad f'_x \left( \frac{\pi}{4}, \frac{\pi}{4} \right) = \frac{1}{2} = 0.5 \\
  f'_y (x, y) &= \frac{\partial f}{\partial y} (x, y) = \frac{2 \sin y \cos y}{2 \sqrt{\sin^2 x + \sin^2 y}} = \frac{\sin y \cos y}{\sqrt{\sin^2 x + \sin^2 y}} \\
  f'_y \left( \frac{\pi}{4}, 0 \right) &= 0; \quad f'_y \left( \frac{\pi}{4}, \frac{\pi}{4} \right) = \frac{1}{2} = 0.5.
\end{align*}
\]

We can also compute these first partial derivatives of the function \( f \) in Matlab 7.9:

\[
\begin{align*}
  >> \text{syms } x \ y \\
  >> f=@(x,y)sqrt(sin(x)^2+sin(y)^2); \quad \text{syms } x \ y \\
  >> s=\text{diff}(f(x,y),x); \quad \text{syms } x \ y \\
  >> \text{ss}\text{=}\text{subs}(s,\{x,y\},\{\pi/4,0\}) \quad \text{syms } x \ y \\
  \text{ss } = \quad 0.7071 \quad \text{syms } x \ y \\
  >> s1\text{=}\text{subs}(s,\{x,y\},\{\pi/4,\pi/4\}) \quad \text{syms } x \ y
\end{align*}
\]