7
Equations and Systems of Linear Ordinary Differential Equations

7.1 Successive Approximation Method

Definition 7.1 (see [15], p. 292). A Cauchy problem consists in a differential equation

\[ y' = f(x, y), \quad f : D \to \mathbb{R}, \quad D \subseteq \mathbb{R}^2 \]

and an initial condition:

\[ y(x_0) = y_0. \]

Let be the Cauchy problem:

\[
\begin{aligned}
    y' &= f(x, y) \\
    y(x_0) &= y_0.
\end{aligned}
\]  

(7.1)

The solution of the problem (7.1) can be found as a limit of the sequence of functions defined by the successive approximation method, (see [45], p. 77):

\[
y_n(x_0) = y_0 + \int_{x_0}^x f(t, y_{n-1}(t)) \, dt, \quad n = 1, 2 \ldots
\]  

(7.2)

Example 7.2. Solve the Cauchy problem:

\[
\begin{aligned}
    y' &= xy \\
    y(0) &= 1
\end{aligned}
\]
using the successive approximation method.

Solution.
The sequence of the successive approximation method (7.2) is:

\[ y_0(x) = 1 \]
\[ y_1(x) = 1 + \int_0^x t \left(1 + \frac{t^2}{2}\right) dt = 1 + \int_0^x t \left(1 + \frac{x^2}{2}ight) = 1 + \frac{x^2}{1!} \]
\[ y_2(x) = 1 + \int_0^x t \left(1 + \frac{t^2}{2}\right) dt = 1 + \int_0^x t \left(1 + \frac{x^2}{2}ight) + \int_0^x \frac{x^4}{8} = 1 + \frac{x^2}{1!} + \frac{(\frac{x^2}{2})^2}{2!} \]
\[ y_3(x) = 1 + \int_0^x t \left(1 + \frac{t^2}{2} + \frac{t^4}{8}\right) dt = 1 + \int_0^x t \left(1 + \frac{x^2}{2} + \frac{x^4}{8}ight) = 1 + \frac{x^2}{1!} + \frac{(\frac{x^2}{2})^2}{2!} + \frac{(\frac{x^2}{2})^3}{3!} \]

\[ y_n(x) = \sum_{k=0}^{n} \frac{(\frac{x^2}{2})^k}{k!}. \]

The solution of our Cauchy problem will be

\[ y(x) = \lim_{n \to \infty} y_n(x) = \sum_{k=0}^{\infty} \frac{(\frac{x^2}{2})^k}{k!} = e^{\frac{x^2}{2}}. \]

We shall also solve this problem in Matlab 7.9:

```matlab
function w=f(u,v)
w= u*v;
end
function r=y(n,t,x,y0)
r=y0;
for k=1:n
    r1=int(f(t,r),t,0,x);
    r=subs(r1,x,t)+y0;
end
end
```

In the command line we shall write: