CHAPTER 3

Patterned Linear Maps

The word “pattern” is employed in common speech to describe almost any organization of elements that exhibits some degree of non-randomness; however, in our context, the term patterned is given a more precise and narrow meaning. In this chapter, we begin by defining a patterned matrix. Our definition relies upon the identification of a base matrix or base pattern and states that the set of all patterned matrices, with respect to a given base, is simply the set of all polynomials of the base matrix. The eigenvalues and eigenvectors of patterned matrices have some notable features, which we examine. An important observation is that eigenvectors of a base matrix are also eigenvectors of all the patterned matrices generated from that base.

The definition of patterned matrices is then extended to the more general concept of patterned linear maps. We explore the relationship between the invariance of a subspace with respect to a base pattern and the invariance of the same subspace with respect to any patterned map. These invariance properties support several important results. First, we obtain that the kernel, image and eigenspaces of patterned maps are invariant with respect to the base transformation. Second, a family of patterned maps can all be decomposed by a common transformation into a set of smaller patterned maps.

The material in this chapter lays the foundation for Chapter 4, where several patterned maps are combined to represent a patterned system in state space form.

3.1 Patterned Matrices

Let $t_0, t_1, \ldots, t_k \in \mathbb{R}$ and consider the polynomial

$$\rho(s) = t_0 + t_1 s + t_2 s^2 + t_3 s^3 + \ldots + t_k s^k.$$
The argument of the polynomial can be extended to become a matrix as follows. Let \( M \) be an \( n \times n \) real matrix. Then \( \rho(M) \) is defined by

\[
\rho(M) := t_0I + t_1M + t_2M^2 + t_3M^3 + \ldots + t_kM^k.
\]

Given \( T = \rho(M) \), then \( \rho(s) \) is called a representor of \( T \) with respect to \( M \), and it is generally not unique. Now suppose that \( k \geq n \). By Cayley-Hamilton it is known that \( M^n \) can always be expressed as a polynomial of lower order powers of \( M \); therefore, our discussion can be confined to \( \rho(M) \) of order less than or equal to \( n - 1 \) without loss of generality. We define the set of all matrices that can be expressed as a polynomial function of a given base matrix \( M \in \mathbb{R}^{n \times n} \) by

\[
\mathfrak{F}(M) := \{ T \mid (\exists t_0, \ldots, t_{n-1} \in \mathbb{R}) \ T = t_0I + t_1M + t_2M^2 + \ldots + t_{n-1}M^{n-1} \}.
\]

We call a matrix \( T \in \mathfrak{F}(M) \) an \( M \)-patterned matrix.

**Remark 3.1.** The set \( \mathfrak{F}(M) \) also encompasses certain infinite polynomials of \( M \) and, more generally, certain functions of \( M \) that have an infinite series representation. The determination of whether a given function of \( M \) is a member of \( \mathfrak{F}(M) \) depends on the analyticity properties of \( \sigma(M) \). Refer to [24] and [3] for a more detailed exposition on this relationship.

For a given base matrix \( M \), all matrices in the class \( \mathfrak{F}(M) \) share a number of useful properties. As a first observation, we note that since all the members of the class can be expressed as polynomials of a common matrix, it follows that all members must commute.

**Lemma 3.2.** Given \( T, R \in \mathfrak{F}(M) \) then \( TR = RT \).

**Remark 3.3.** By Lemma 3.2, all matrices in the set \( \mathfrak{F}(M) \) commute; however, it does not generally follow that every matrix that commutes with \( M \) is also in the set \( \mathfrak{F}(M) \). In our main results on patterned linear systems in Chapters 4 and 5 it is sometimes possible to show simpler proofs by exploiting the commuting property of \( M \)-patterned matrices. In these cases we present the alternate proof in a remark. But, we avoid exploiting commutativity for our general results for two reasons. First, we have not found any method by which problems such as the Patterned Output Stabilization Problem and the Patterned Restricted Regulator Problem can be solved by exploiting commutativity. Second, the system matrices of block patterned systems, our most important future research direction, generally will not commute. Thus, we adopt a framework based on common invariant subspaces, because this appears to be applicable to the widest variety of patterned problems.

It can be further deduced from the form of an \( M \)-patterned matrix that an eigenvector of \( M \) is an eigenvector of every \( M \)-patterned matrix. This is perhaps the most remarkable property of matrices in this class, and it will be shown to have great significance for patterned systems. We explore this