Secondary Splitting of Zero Gradient Points in Turbulent Scalar Fields

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Abstract. The mechanisms related to the secondary splitting of zero gradient points of scalar fields are analyzed using the two dimensional case of a scalar extreme point lying in a region of local strain. The velocity field is assumed to resemble a stagnation point flow, cf. Gibson [1], which is approximated using a Taylor expansion up to third order. It is found that the splitting can only be explained when the third order terms of the Taylor expansion of the flow field are included. The non-dimensional splitting time turns out to depend on three parameters, namely the local Péclet number Pe_δ based on the initial size of the extreme point δ and two parameters which are measures of the rate of change of the local strain. For the limiting case Pe_δ → 0, the splitting time is found to be finite but Péclet number independant, while for the case of Pe_δ → ∞ it increases logarithmically with the Péclet number.

1 Introduction

Gibson [1] analyzed the behaviour at the smallest scales of turbulent scalar fields in terms of the properties of zero gradient points and minimal gradient surfaces. He concluded that these regions of the field are of physical importance to the problem of turbulent mixing. Among other findings, Gibson identified two physical mechanisms which lead to the creation of new zero gradient points: while initially zero gradient points must be created from regions of uniform scalar gradient, he concludes that the majority of such points results from the combined action of strain and diffusion on existing zero gradient points, which leads to their splitting into new extreme points. This effect is called secondary splitting. In recent works by Peters et al. [2][3][4], extreme points of turbulent scalar fields have again received attention in the context of dissipation element theory. The temporal evolution of dissipation elements is inherently connected to the evolution of their ending points.
In the present study we will consider the splitting to take place in a two-dimensional environment to extract the relevant physical mechanisms and their interplay. In a coordinate system moving with the extremum the flow field relative to the extreme point can be described as a stagnation point flow. Mathematically, the problem is described by the transport equation for the passive scalar \( \phi \), which reads

\[
\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} = D \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right).
\]

In Eq. (1) \( \phi \) denotes the two-dimensional scalar field, and \( u \) and \( v \) are the velocity components of the velocity field relative to the extreme point in \( x \)– and \( y \)–direction. \( D \) denotes the constant diffusivity of the scalar. We restrict the analysis to a symmetric problem for the scalar as well as the velocity field. An extreme point of finite size can be described locally by a Gaussian bell-shaped-curve of the form

\[
\phi(x,y,t_0) = \phi_0 \exp\left(\frac{(-x^2 + y^2)}{\delta^2}\right),
\]

where the sign of \( \phi_0 \) determines the type of the extreme point while \( \delta \) determines its characteristic size. The stagnation point flow around the extreme point will be expanded in a Taylor series up to third order. We introduce

\[
\alpha = \frac{\partial u}{\partial x}|_{x_0,y_0} = -\frac{\partial v}{\partial y}|_{x_0,y_0}, \quad \beta_x = \frac{\partial^3 u}{\partial x^3}|_{x_0,y_0}, \quad \beta_y = \frac{\partial^3 v}{\partial y^3}|_{x_0,y_0},
\]

and non-dimensionalize the problem using the variables \( \tilde{\phi} = \phi/\phi_0, \tilde{t} = \alpha t, \tilde{x} = x/\delta \) and \( \tilde{y} = y/\delta \). Dropping the tildes for simplicity, the problem reads

\[
\frac{\partial \phi}{\partial t} + \left( x + \frac{1}{6} B_x x^3 - \frac{1}{2} B_y x^2 y \right) \frac{\partial \phi}{\partial x} + \left( -y + \frac{1}{6} B_y y^3 - \frac{1}{2} B_x x^2 y \right) \frac{\partial \phi}{\partial y} = \frac{1}{Pe_\delta} \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right),
\]

with the non-dimensional parameters \( Pe_\delta = (\alpha \delta^2)/D \), \( B_x = (\delta^2 \beta_x)/\alpha \) and \( B_y = (\delta^2 \beta_y)\alpha \). While the first of these is a Péclet number based on the characteristic size \( \delta \) of the extreme point, the latter two are measures for the rate of change of the local strain in \( x \)– and \( y \)–direction.

### 2 Analysis and Validation

As the problem stated in Eq. (3) is entirely symmetric, one can argue that the splitting will take place at the symmetry-point, which can be shifted to the origin without loss of generality. The splitting time is then characterized by the time at which the second derivative of the scalar field at the origin changes its sign. As an analytical solution of Eq. (3) is not readily available, we seek a solution by expanding \( \phi \) in terms of a series of the form