Modal and Topological Operators Defined over IFSs

4.1 “Necessity” and “Possibility” Operators Defined over IFSs

This Chapter includes several operators over IFSs which have no counterparts in the ordinary fuzzy set theory.

Initially, following [11, 39], we introduce two operators over IFSs that transform an IFS into a fuzzy set (i.e., a particular case of an IFS). They are similar to the operators “necessity” and “possibility” defined in some modal logics. Their properties resemble those of the modal logic (see e.g. [227]).

Let, for every IFS $A$,

$$\Box A = \{ \langle x, \mu_A(x), 1 - \mu_A(x) \rangle | x \in E \}.$$  \hspace{1cm} (4.1)

$$\Diamond A = \{ \langle x, 1 - \nu_A(x), \nu_A(x) \rangle | x \in E \}. $$  \hspace{1cm} (4.2)

Obviously, if $A$ is an ordinary fuzzy set, then

$$\Box A = A = \Diamond A. $$  \hspace{1cm} (4.3)

The equalities (4.3) show that these operators do not have analogues in the case of fuzzy sets; and therefore, this is a new demonstration of the fact that IFSs are proper extensions of the ordinary fuzzy sets.

Most of the results described so far were, to some extent, similar to these of the ordinary fuzzy set theory. It was the definition of the two new operators that made it clear that IFSs are objects of different nature.

Both operators (4.1) and (4.2) were defined in March 1983 and this stimulated the author to continue his research on IFSs. At that moment George Gargov named the new sets “Intuitionistic Fuzzy Sets”.

The following paragraphs consider the properties, modifications and extensions of these new operators.

For every IFS $A$,
(a) \[\Box A = \Diamond A,\]
(b) \[\Diamond A = \Box A,\]
(c) \[\Box A \subseteq A \subseteq \Diamond A,\]
(d) \[\Box \Box A = \Box A,\]
(e) \[\Box \Diamond A = \Diamond A,\]
(f) \[\Diamond \Box A = \Box A,\]
(g) \[\Diamond \Diamond A = \Diamond A.\]

For example, the validity of (a) is checked as follows:

\[\Box A = \Box \{\langle x, \nu_A(x), \mu_A(x) \rangle | x \in E\} = \Box \{\langle x, 1 - \nu_A(x), \nu_A(x) \rangle | x \in E\} = \Diamond A.\]

For every two IFSs \(A\) and \(B\),

(a) \[\Box (A \cap B) = \Box A \cap \Box B,\]
(b) \[\Box (A \cup B) = \Box A \cup \Box B,\]
(c) \[\Box (A + B) = \Box A + \Box B,\]
(d) \[\Box (A.B) = \Box A \cdot \Box B,\]
(e) \[\Box (A + B) = \Diamond A \cdot \Diamond B,\]
(f) \[\Box (A.B) = \Diamond A \cdot \Diamond B,\]
(g) \[\Box (A @ B) = \Box A @ \Box B,\]
(h) \[\Diamond (A \cap B) = \Diamond A \cap \Diamond B,\]
(i) \[\Diamond (A \cup B) = \Diamond A \cup \Diamond B,\]
(j) \[\Diamond (A + B) = \Diamond A + \Diamond B,\]
(k) \[\Diamond (A.B) = \Diamond A \cdot \Diamond B,\]
(l) \[\Diamond (A + B) = \Box A \cdot \Box B,\]
(m) \[\Diamond (A.B) = \Box A + \Box B,\]
(n) \[\Diamond (A @ B) = \Diamond A @ \Diamond B,\]
(o) \[\Box (\bigcap_{i=1}^{n} A_i) = \bigcap_{i=1}^{n} (\Box A_i),\]
(p) \[\Diamond (\bigcap_{i=1}^{n} A_i) = \bigcap_{i=1}^{n} (\Diamond A_i).\]

Two new relations are defined as follows:

\[A \subseteq \Box B \text{ iff } (\forall x \in E)(\mu_A(x) \leq \mu_B(x)),\]
\[A \subseteq \Diamond B \text{ iff } (\forall x \in E)(\nu_A(x) \geq \nu_B(x)).\]