Extended Modal Operators

5.1 Operators $D_\alpha$ and $F_{\alpha,\beta}$

Following [18, 19, 39], we construct an operator which represents both operators $\Box$ from (4.1) and $\Diamond$ from (4.2). It has no analogue in the ordinary modal logic, but the author hopes that the search for such an analogue in modal logic will be interesting.

Let $\alpha \in [0, 1]$ be a fixed number. Given an IFS $A$, we define an operator $D_\alpha$ as follows:

$$D_\alpha(A) = \{ \langle x, \mu_A(x) + \alpha \pi_A(x), \nu_A(x) + (1 - \alpha) \pi_A(x) \rangle | x \in E \}.$$  \hspace{1cm} (5.1)

From this definition it follows that $D_\alpha(A)$ is a fuzzy set, because:

$$\mu_A(x) + \alpha \pi_A(x) + \nu_A(x) + (1 - \alpha) \pi_A(x) = \mu_A(x) + \nu_A(x) + \pi_A(x) = 1.$$  

Some of the specific properties of this operator are:

(a) if $\alpha \leq \beta$, then $D_\alpha(A) \subseteq D_\beta(A)$;
(b) $D_0(A) = \Box A$;
(c) $D_1(A) = \Diamond A,$

for every IFS $A$ and for every $\alpha, \beta \in [0, 1]$.

To every point $x \in E$ the operator $f_{D_\alpha}(A)$ assigns a point of the segment between $f_{\Box} A(x)$ and $f_{\Diamond} A(x)$ depending on the value of the argument $\alpha \in [0, 1]$ (see Fig. 5.1). As in the case of some of the above operations, this construction needs auxiliary elements which are shown in Fig. 5.1.

As we noted above, the operator $D_\alpha$ is an extension of the operators $\Box$ and $\Diamond$, but it can be extended even further.

Let $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Define (see [18, 19, 39]) the operator $F_{\alpha,\beta}$, for the IFS $A$, by

$$F_{\alpha,\beta}(A) = \{ \langle x, \mu_A(x) + \alpha \pi_A(x), \nu_A(x) + \beta \pi_A(x) \rangle | x \in E \}.$$  \hspace{1cm} (5.2)
For every IFS $A$, and for every $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta \leq 1$,

(a) $F_{\alpha, \beta}(A)$ is an IFS;
(b) if $0 \leq \gamma \leq \alpha$, then $F_{\gamma, \beta}(A) \subseteq F_{\alpha, \beta}(A)$;
(c) if $0 \leq \gamma \leq \beta$, then $F_{\alpha, \beta}(A) \subseteq F_{\alpha, \gamma}(A)$;
(d) $D_{\alpha}(A) = F_{\alpha, 1 - \alpha}(A)$;
(e) $\Box A = F_{0, 1}(A)$;
(f) $\Diamond A = F_{1, 0}(A)$;
(g) $F_{\alpha, \beta}(A) = F_{\beta, \alpha}(A)$
(h) $\mathcal{C}(F_{\alpha, \beta}(A)) \subseteq F_{\alpha, \beta}\mathcal{C}(A)$,
(i) $\mathcal{I}(F_{\alpha, \beta}(A)) \supseteq F_{\alpha, \beta}\mathcal{I}(A)$.

Let us prove property (h):

$$
\mathcal{C}(F_{\alpha, \beta}(A)) = \mathcal{C}((x, \mu_A(x) + \alpha_\pi_A(x), \nu_A(x) + \beta_\pi_A(x))|x \in E})
= \{(x, K_1, L_1)|x \in E\},
$$

where

$$
K_1 = \sup_{y \in E}(\mu_A(y) + \alpha_\pi_A(y)),
$$

$$
L_1 = \inf_{y \in E}(\nu_A(y) + \beta_\pi_A(y)),
$$

and

$$
F_{\alpha, \beta}(\mathcal{C}(A)) = F_{\alpha, \beta}(\{(x, K, L)|x \in E\})
= \{(x, K + \alpha_\beta(1 - K - L), L + \beta_\beta(1 - K - L))|x \in E\},
$$

where $K$ and $L$ are defined by (4.8) and (4.9).