Norms and Metrics over IFSs or Their Elements

7.1 Standard IF-Norms and Metrics

First, let us emphasize that here we do not study the usual set-theoretic properties of the IFSs (i.e. properties which follow directly from the fact that IFSs are sets in the sense of the set theory – see Section 2.1). For example, given a metric space $E$, one can study the metric properties of the IFSs over $E$. This can be done directly by topological methods (see e.g. [421]) and the essential properties of the IFSs are not used. On the other hand, all IFSs (and hence, all fuzzy sets) over a fixed universe $E$ generate a metric space (in the sense of [421]), but with a special metric (cf., e.g., [301]): one that is not related to the elements of $E$ and to the values of the functions $\mu_A$ and $\nu_A$ defined for these elements.

This peculiarity is based on the fact that the “norm” of a given IFS’ element is not actually a norm in the sense of [421]. Rather, it is in some sense a “pseudo-norm” which assigns a number to every element $x \in E$. This number depends on the values of the functions $\mu_A$ and $\nu_A$ (which are calculated for this element).

Thus, the important conditions,

$$\|x\| = 0 \text{ iff } x = 0,$$

and

$$\|x\| = \|y\| \text{ iff } x = y,$$

do not hold here.

Instead, the following is valid

$$\|x\| = \|y\| \text{ iff } \mu_A(x) = \mu_A(y) \text{ and } \nu_A(x) = \nu_A(y).$$

Actually, the value of $\mu_A(x)$ plays the role of a norm (more precisely, a pseudo-norm) for the element $x \in E$ in every fuzzy set over $E$. In the intuitionistic fuzzy case, the existence of the second functional component – the function $\nu_A$ – gives rise to different options for the definition of the
concept of norm (in the sense of pseudo-norm) over the subsets and the elements of a given universe $E$.

The first intuitionistic fuzzy norm for every $x \in E$ with respect to a fixed set $A \subseteq E$ is

$$
\sigma_{1,A}(x) = \mu_A(x) + \nu_A(x).
$$

(7.1)

It represents the degree of definiteness of the element $x$. From

$$
\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)
$$

it follows that

$$
\sigma_{1,A}(x) = 1 - \pi_A(x).
$$

For every two IFSs $A$ and $B$, and for every $x \in E$,

(1.1) $\sigma_{1,A}(x) = \sigma_{1,A}(x)$;
(1.2) $\sigma_{1,A \cap B}(x) \geq \min(\sigma_{1,A}(x), \sigma_{1,B}(x))$;
(1.3) $\sigma_{1,A \cup B}(x) \leq \max(\sigma_{1,A}(x), \sigma_{1,B}(x))$;
(1.4) $\sigma_{1,A + B}(x) \geq \sigma_{1,A}(x) \cdot \sigma_{1,B}(x)$;
(1.5) $\sigma_{1,A \cdot B}(x) \geq \sigma_{1,A}(x) \cdot \sigma_{1,B}(x)$;
(1.6) $\sigma_{1,A \oplus B}(x) = \frac{(\sigma_{1,A}(x) + \sigma_{1,B}(x))}{2}$;
(1.7) $\sigma_{1,\Box A}(x) = 1$;
(1.8) $\sigma_{1,\Diamond A}(x) = 1$;
(1.9) $\sigma_{1,\text{c}(A)}(x) \geq \max_{x \in E} \sigma_{1,A}(x)$;
(1.10) $\sigma_{1,\text{z}(A)}(x) \leq \min_{x \in E} \sigma_{1,A}(x)$;
(1.11) $\sigma_{1,D_{\alpha}(A)}(x) = 1$, for every $\alpha \in [0, 1]$;
(1.12) $\sigma_{1,F_{\alpha,\beta}(A)}(x) = \alpha + \beta + (1 - \alpha - \beta) \cdot \sigma_{1,A}(x)$ for every $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$;
(1.13) $\sigma_{1,G_{\alpha,\beta}(A)}(x) \leq \sigma_{1,A}(x)$, for every $\alpha, \beta \in [0, 1]$;
(1.14) $\sigma_{1,H_{\alpha,\beta}(A)}(x) \leq \beta + (\alpha + \beta) \cdot \sigma_{1,A}(x)$, for every $\alpha, \beta \in [0, 1]$;
(1.15) $\sigma_{1,H_{\alpha,\beta}(A)}(x) \leq \beta + (1 - \beta) \cdot \sigma_{1,A}(x)$, for every $\alpha, \beta \in [0, 1]$;
(1.16) $\sigma_{1,J_{\alpha,\beta}(A)}(x) \leq \alpha + (\alpha + \beta) \cdot \sigma_{1,A}(x)$, for every $\alpha, \beta \in [0, 1]$;
(1.17) $\sigma_{1,J_{\alpha,\beta}(A)}(x) \leq \alpha + (1 - \alpha) \cdot \sigma_{1,A}(x)$, for every $\alpha, \beta \in [0, 1]$;
(1.18) $\sigma_{1,X_{\alpha,b,c,d,e,f}(A)}(x) \begin{cases} 
\leq \max(a - b - ef, d - e - bc) + b + e \\
\geq \min(a - b - ef, d - e - bc) + b + e
\end{cases}$;
(1.19) $\sigma_{1,P_{\alpha,\beta}(A)}(x) \geq \alpha$, for every $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$;
(1.20) $\sigma_{1,Q_{\alpha,\beta}(A)}(x) \geq \beta$, for every $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$;
(1.21) $\sigma_{1,\Box A}(x) = \frac{1}{2} \sigma_{1,A}(x) + \frac{1}{2}$.