Spatial Geometry and Vector Algebra

vector, a quantity having direction as well as magnitude (from Latin vector, carrier; stems from vehere, to carry)

(Hoad, *The Concise Oxford Dictionary of English Etymology*)

Spatial geometry starts in Book XI of Euclid’s *Elements* with a long list of definitions and boring propositions. It is here that analytic geometry shows its full power: one simply adds a third coordinate \( z \). If you know how to calculate with two variables, you can also calculate with three.

Just as easily, one then adds a fourth coordinate, then a fifth one, etc. The only constraint is the limited supply of letters. It is thus judicious to write

\[ x_1, x_2, \ldots, x_n \]  

(9.1)

for the coordinates.

**Vector notation.** A second revolution, comparable to that of Descartes, took place towards the end of the 19th century with the introduction of vectors.

At that time, one began to consider \( n \)-tuples of coordinates as *new mathematical objects*\(^1\)

\[ x = (x_1, x_2, \ldots, x_n) \quad \text{or} \quad a = (a_1, a_2, \ldots, a_n). \]  

(9.2)

With these objects, one can perform *algebraic operations*, such as calculating their sum and their product with a scalar, by performing them componentwise:

\[ a + b = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n), \quad \lambda a = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n). \]  

(9.3)

The vector notation results in much shorter and clearer proofs. Moreover, the proofs are the same for *any dimension.*

\(^1\)Many authors distinguish vectors from scalars by a special notation. For example, vectors are often denoted as \( \mathbf{a}, \mathbf{x} \) or \( \overrightarrow{a}, \overrightarrow{x} \) or \( \vec{a}, \vec{x} \) or \( \dot{a}, \dot{x} \) or \( a, x \). Like Banach (see Fig. 9.2) we use ordinary letters in the following.
Historical development of vectors. The introduction of vectors can be traced back to several origins.

(a) Grassmann (extensive Größen [extensive quantities]).

“... il est très utile d’introduire la considération des nombres complexes, ou nombres formés avec plusieurs unités, ... [it is very useful to introduce the consideration of complex numbers, or numbers composed of several quantities]”


“... e il lavoro più profondo che abbia su questo soggetto è senza dubbio l’Ausdehnungslehre pubblicato dal Grassmann ... [and the profoundest work which we have on this subject is without doubt the Ausdehnungslehre published by Grassmann]”

(Peano 1894, Opere Scelte III, p. 340)

The German theologian and linguist Hermann Grassmann (1809–1877), self-taught in mathematics, published in 1844 his work Die lineale Ausdehnungslehre, an unreadable book, interspersed with mystic and abstract considerations. In 1862, a revised edition appeared, which did not attract more attention. Grassmann’s ideas became more widely known in mathematics only 20 to 30 years later (see the quotations from Peano).

(b) Hamilton (quaternions).

“At the age of five Hamilton could read Latin, Greek, and Hebrew. At eight he added Italian and French; at ten he could read Arabic and Sanskrit and at fourteen, Persian.”

(M. Kline, 1972, p. 777)

“Tomorrow will be the fifteenth birthday of the Quaternions. They started into life, or light, full grown, on the 16th of October, 1843, as I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge. That is to say, I then and there felt the galvanic circuit of thought closed ... I felt a problem to have been at that moment solved, an intellectual want relieved, which had haunted me for at least fifteen years before.”

(Hamilton; quoted by M. Kline, 1972, p. 779)

In 1837, William R. Hamilton (1805–1865), a celebrated Irish physicist (optics, mechanics) and mathematician, introduced the complex numbers

\[ a + ib \leftrightarrow (a, b) \]

as pairs of real numbers. This definition is still used today. Later, he struggled mightily but unsuccessfully (see the quotation) to generalise these numbers, which can be multiplied and divided, to three dimensions. Finally, in 1843 he found a generalisation to four dimensions

\[ a + ib + jc + kd , \]

the quaternions. With the noncommutative multiplication rules