NE Is Not NP Turing Reducible to Nonexponentially Dense NP Sets

Bin Fu

Department of Computer Science
University of Texas-Pan American, Edinburg, TX 78539, USA
binfu@cs.panam.edu

Abstract. A long standing open problem in the computational complexity theory is to separate NE from BPP, which is a subclass of \( \text{NP}_T(\text{NP}) \cap \text{P}/\text{Poly} \). In this paper, we show that \( \text{NE} \not\subseteq \text{NP}_T(\text{NP} \cap \text{Nonexponentially-Dense-Class}) \), where Nonexponentially-Dense-Class is the class of languages \( A \) without exponential density (for each constant \( c > 0 \), \( |A| \leq 2^{n^c} \) for infinitely many integers \( n \)). Our result implies \( \text{NE} \not\subseteq \text{NP}_T(\text{padding}(\text{NP}, g(n))) \) for every time constructible super-polynomial function \( g(n) \) such as \( g(n) = n^{\lceil \log \lceil \log n \rceil \rceil} \), where Padding(\( \text{NP} \), \( g(n) \)) is class of all languages \( L_B = \{ s10^{g(|s|)}-|s|-1 : s \in B \} \) for \( B \in \text{NP} \). We also show \( \text{NE} \not\subseteq \text{NP}_T(\text{P}_{tt}(\text{NP}) \cap \text{TALLY}) \).

1 Introduction

Separating the complexity classes has been one of the central problems in complexity theory. Separating \( \text{NEXP} \) from \( \text{P/Pol} \) is a long standing fundamental open problem in the computational complexity theory. We do not even know how to separate \( \text{NEXP} \) from \( \text{BPP} \), which is a subclass of \( \text{NP}_T(\text{NP} \cap \text{Nonexponentially-Dense-Class}) \), where Nonexponentially-Dense-Class is the class of languages \( A \) without exponential density (for each constant \( c > 0 \), \( |A| \leq 2^{n^c} \) for infinitely many integers \( n \)).

Whether sparse sets are hard for complexity classes plays an important role in the computational complexity theory (for examples, [3][6][13][14]). It is well known that \( \text{P}/\text{Pol} \) is the same as the class of languages that are truth table reducible to tally sets (\( \text{P}/\text{Pol} = \text{P}_{tt}(\text{TALLY}) \)). The combination of bounded number of queries and density provides an approach to characterize the complexity of the nonuniform computation models. The partial progress for separating exponential time classes from nonuniform polynomial time classes are shown in [20][7][10][12][15]. Let Nonexponentially-Dense-Class be the class of languages \( A \) without exponential density (for each constant \( c > 0 \), \( |A| \leq 2^{n^c} \) for infinitely many integers \( n \)). Improving Hartmanis and Berman’s separation \( E \not\subseteq \text{P}_m(\text{Nonexponentially-Dense-Class}) \) [3], Watanabe [20] showed \( E \not\subseteq \text{P}_{tt}(\text{Nonexponentially-Dense-Class}) \). Watanabe’s result was improved by two research groups independently with incomparable results that \( E \not\subseteq \text{P}_{n^1-\epsilon-\text{tt}}(\text{Nonexponentially-Dense-Class}) \) by Lutz and Mayordomo [15], and \( \text{EXP} \not\subseteq \text{P}_{n^1-\epsilon-\text{T}}(\text{Nonexponentially-Dense-Class}) \) and \( E \not\subseteq \text{P}_{n^{1-\epsilon-\text{T}}}(\text{Nonexponentially-Dense-Class}) \) by Fu [7]. Fu’s results were improved.
to E \not\subseteq P_{n^c-T}(\text{Nonexponentially-Dense-Class}) by Hitchcock [12]. A recent celebrated progress was made by Williams separating NEXP from ACC [21]. It is still an open problem to separate NEXP from P_{O(n)-tt}(TALLY).

The nondeterministic time hierarchy was separated in the early research of complexity theory by Cook [6], Serferas, Fischer, Meyer [10], and Zak [22]. A separation with immunity among nondeterministic computational complexity classes was derived by Allender, Beigel, Hertrampf and Homer [2]. The difference between NE and NP has not been fully solved. One of the most interesting problems between them is to separate NE from P_T(NP). Fu, Li and Zhong [9] showed NE \not\subseteq P_{n^{o(1)}-T}(NP). Their result was later improved by Mocas [17] to NEXP \not\subseteq P_{n^{c-T}}(NP) for any constant c > 0. Mocas’s result is optimal with respect to relativizable proofs, as Buhrman and Torenvliet [5] showed an oracle for which NEXP = P_T(NP). Buhrman, Fortnow and Santhanam [4] and Fu, Li and Zhang [8] showed NEXP \not\subseteq P_{n^{c-T}}(NP)/n^c for every constant c > 0 (two papers appeared in two conferences with a similar time). Fu, Li and Zhang showed that NEXP is not reducible to tally sets by the polynomial time nondeterministic Turing reductions with the number of queries bounded by a sub-polynomial function g(n) such as g(n) = n^{\frac{\log \log n}{c}} (NE \not\subseteq NP g(n)-T(TALLY)) [8].

In this paper, we show that NE \not\subseteq NP_T(NP \cap \text{Nonexponentially-Dense-Class}). Our result implies NE \not\subseteq NP_T(\text{padding}(NP, g(n))) for every time constructible super-polynomial function g(n) such as g(n) = n^{\log\log n}, where Padding(NP, g(n)) is the class of all languages \{s10^{|s|}-|s|-1 : s \in B\} for B \in NP. We also show NE \not\subseteq NP_T(P_{tt}(NP) \cap TALLY).

This paper is organized as follows. Some notations are given in section 2. In section 3 we give a brief description of our method to prove the main result. In section 4 we separate NE from NP_T(NP \cap \text{Nonexponentially-Dense-Class}). In section 5 we show how to use the padding method to derive sub-exponential density problems in the class NP. The conclusions are given in section 6.

2 Notations

Let N = \{0, 1, 2, \ldots\} be the set of all natural numbers. Let \Sigma = \{0, 1\} be the alphabet for all the languages in this paper. The length of a string s is denoted by |s|. Let A be a language. A^{\leq n} is the subset of strings of length at most n in A. A^{=n} is the subset of strings of length n in A. For a finite set X, let |X| be the number of elements in X. For a Turing machine M(.), let L(M) be the language accepted by M. We use a pairing function (.,.) with |(x, y)| = O(|x| + |y|).

For a function t(n) : N \rightarrow N, let DTIME(t(n)) be the class of languages accepted by deterministic Turing machines in O(t(n)) time, and NTIME(t(n)) be the class of languages accepted by nondeterministic Turing machines in O(t(n)) time. Define the exponential time complexity classes: E = \bigcup_{c=1}^{\infty} \text{DTIME}(2^{cn}), \text{EXP} = \bigcup_{c=1}^{\infty} \text{DTIME}(2^{cn}), \text{NE} = \bigcup_{c=1}^{\infty} \text{NTIME}(2^{cn}) and NEXP = \bigcup_{c=1}^{\infty} \text{NTIME}(2^{2^cn})

A language L is sparse if for some constant c > 0, |L^{\leq n}| \leq n^c for all large n. Let SPARSE represent all sparse languages. Let TALLY be the class of languages with alphabet \{1\}. 