NE Is Not NP Turing Reducible to Nonexponentially Dense NP Sets

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Abstract. A long standing open problem in the computational complexity theory is to separate NE from BPP, which is a subclass of $\text{NP}_T(\text{NP}) \cap \text{P}/\text{Poly}$. In this paper, we show that $\text{NE} \nsubseteq \text{NP}_T(\text{NP} \cap \text{Nonexponentially-Dense-Class})$, where Nonexponentially-Dense-Class is the class of languages $A$ without exponential density (for each constant $c > 0$, $|A| \leq 2^{n^c}$ for infinitely many integers $n$). Our result implies $\text{NE} \nsubseteq \text{NP}_T(\text{padding}(\text{NP}, g(n)))$ for every time constructible superpolynomial function $g(n)$ such as $g(n) = n^{\log \log n}$, where $\text{Padding}(\text{NP}, g(n))$ is class of all languages $L_B = \{s10^{g(|s|)} - |s| - 1 : s \in B\}$ for $B \in \text{NP}$. We also show $\text{NE} \nsubseteq \text{NP}_T(\text{P}_{\text{tt}}(\text{NP}) \cap \text{TALLY})$.

1 Introduction

Separating the complexity classes has been one of the central problems in complexity theory. Separating $\text{NEXP}$ from $\text{P}/\text{Poly}$ is a long standing fundamental open problem in the computational complexity theory. We do not even know how to separate $\text{NEXP}$ from $\text{BPP}$, which is a subclass of $\text{NP}_T(\text{NP}) \cap \text{P}/\text{Poly}$ proved by Adleman [1].

Whether sparse sets are hard for complexity classes plays an important role in the computational complexity theory (for examples, [3][16][15][14]). It is well known that $\text{P}/\text{Poly}$ is the same as the class of languages that are truth table reducible to tally sets ($\text{P}/\text{Poly} = \text{P}_{\text{tt}}(\text{TALLY})$). The combination of bounded number of queries and density provides an approach to characterize the complexity of the nonuniform computation models. The partial progress for separating exponential time classes from nonuniform polynomial time classes are shown in [20][7][10][12][15]. Let Nonexponentially-Dense-Class be the class of languages $A$ without exponential density (for each constant $c > 0$, $|A| \leq 2^{n^c}$ for infinitely many integers $n$). Improving Hartmanis and Berman’s separation $\text{E} \nsubseteq \text{P}_m(\text{Nonexponentially-Dense-Class})$ [3], Watanabe [20] showed $\text{E} \nsubseteq \text{P}_{\text{tt}}(\text{Nonexponentially-Dense-Class})$. Watanabe’s result was improved by two research groups independently with incomparable results that $\text{E} \nsubseteq \text{P}_{n^{1-\epsilon}_{\text{-tt}}}(\text{Nonexponentially-Dense-Class})$ by Lutz and Mayordomo [15], and $\text{EXP} \nsubseteq \text{P}_{n^{1-\epsilon}_{\text{-T}}}(\text{Nonexponentially-Dense-Class})$ and $\text{E} \nsubseteq \text{P}_{n^{1-\epsilon}_{\text{-T}}}(\text{Nonexponentially-Dense-Class})$ by Fu [7]. Fu’s results were improved...
to $E \not\subseteq P_{n^c-T}(\text{Nonexponentially-Dense-Class})$ by Hitchcock [12]. A recent celebrated progress was made by Williams separating $\text{NEXP}$ from $\text{ACC}$ [21]. It is still an open problem to separate $\text{NEXP}$ from $P_{O(n^c-T)}(\text{TALLY})$.

The nondeterministic time hierarchy was separated in the early research of complexity theory by Cook [6], Serferas, Fischer, Meyer [19], and Zak [22]. A separation with immunity among nondeterministic computational complexity classes was derived by Allender, Beigel, Hertrampf and Homer [2]. The difference between $\text{NE}$ and $\text{NP}$ has not been fully solved. One of the most interesting problems between them is to separate $\text{NE}$ from $P_T(\text{NP})$. Fu, Li and Zhong [9] showed $\text{NE} \not\subseteq P_{n^c-T}(\text{NP})$. Their result was later improved by Mocas [17] to $\text{NEXP} \not\subseteq P_{n^c-T}(\text{NP})$ for any constant $c > 0$. Mocas’s result is optimal with respect to relativizable proofs, as Buhrman and Torenvliet [5] showed an oracle relative to which $\text{NEXP} = P_T(\text{NP})$. Buhrman, Fortnow and Santhanam [4] and Fu, Li and Zhang [8] showed $\text{NEXP} \not\subseteq P_{n^c-T}(\text{NP})/n^c$ for every constant $c > 0$ (two papers appeared in two conferences with a similar time). Fu, Li and Zhang showed that $\text{NEXP}$ is not reducible to tally sets by the polynomial time nondeterministic Turing reductions with the number of queries bounded by a sub-polynomial function $g(n)$ such as $g(n) = n^{\frac{\log \log n}{n}}$ (NE $\not\subseteq P_{g(n)-T}(\text{TALLY})$) [8].

In this paper, we show that $\text{NE} \not\subseteq P_{T}(\text{NP} \cap \text{Nonexponentially-Dense-Class})$. Our result implies $\text{NE} \not\subseteq P_{T}(\text{padding}(\text{NP}, g(n)))$ for every time constructible super-polynomial function $g(n)$ such as $g(n) = n^{\frac{\log \log n}{n}}$, where $\text{Padding}(\text{NP}, g(n))$ is the class of all languages $L_B = \{s10^{g(|s|)-|s|-1} : s \in B\}$ for $B \in \text{NP}$. We also show $\text{NE} \not\subseteq P_{T}(\text{padding}(\text{NP}) \cap \text{TALLY})$.

This paper is organized as follows. Some notations are given in section 2. In section 3 we give a brief description of our method to prove the main result. In section 4 we separate $\text{NE}$ from $P_{T}(\text{NP} \cap \text{Nonexponentially-Dense-Class})$. In section 5 we show how to use the padding method to derive sub-exponential density problems in the class NP. The conclusions are given in section 6.

2 Notations

Let $N = \{0, 1, 2, \ldots\}$ be the set of all natural numbers. Let $\Sigma = \{0, 1\}$ be the alphabet for all the languages in this paper. The length of a string $s$ is denoted by $|s|$. Let $A$ be a language. $A^{\leq n}$ is the subset of strings of length at most $n$ in $A$. $A^{=n}$ is the subset of strings of length $n$ in $A$. For a finite set $X$, let $|X|$ be the number of elements in $X$. For a Turing machine $M(\cdot)$, let $L(M)$ be the language accepted by $M$. We use a pairing function $(\cdot, \cdot)$ with $|\langle x, y \rangle| = O(|x| + |y|)$.

For a function $t(n) : N \to N$, let $\text{DTIME}(t(n))$ be the class of languages accepted by deterministic Turing machines in $O(t(n))$ time, and $\text{NTIME}(t(n))$ be the class of languages accepted by nondeterministic Turing machines in $O(t(n))$ time. Define the exponential time complexity classes: $E = \bigcup_{c=1}^{\infty} \text{DTIME}(2^{cn})$, $\text{EXP} = \bigcup_{c=1}^{\infty} \text{DTIME}(2^{cn^c})$, $\text{NE} = \bigcup_{c=1}^{\infty} \text{NTIME}(2^{cn})$ and $\text{NEXP} = \bigcup_{c=1}^{\infty} \text{NTIME}(2^{cn^c})$.

A language $L$ is sparse if for some constant $c > 0$, $|L^{\leq n}| \leq n^c$ for all large $n$. Let $\text{SPARSE}$ represent all sparse languages. Let $\text{TALLY}$ be the class of languages with alphabet $\{1\}$.