Chapter 4
Generalized Uncertain Fuzzy Logic Systems

Abstract. In this chapter, basic constructions of fuzzy logic systems with uncertain membership functions are presented. We begin with historical approaches to reasoning with interval-valued fuzzy sets and known formulations of general type-2 fuzzy logic systems. Next we provide new formulations grounded in non-singleton fuzzification. In the context of ordinary fuzzy systems, we demonstrate that variously interpreted non-singleton fuzzification, for typical structures fuzzy logic systems, can be implemented by the classical singleton structures only using modified antecedent fuzzy sets. The first approach to fuzzification of premises is done by the interpretation in terms of possibility distributions of actual inputs. Consequently, the possibility and necessity measures of antecedent fuzzy sets create boundaries for the interval-valued antecedent membership function. The second approach applies rough approximations to antecedent fuzzy sets by non-singleton fuzzy premise sets considered as fuzzy-rough partitions. Two known definitions, the one of Dubois and Prade, and the second proposed by Nakamura, lead to different formulations of fuzzy logic systems. Employing fuzzy-rough sets of Dubois and Prade, we obtain the interval-valued fuzzy logic system. Then, it can be immediately proved that upper approximations in fuzzy-rough systems are concurrent to fuzzification in conjunction-type fuzzy systems. Unexpectedly, lower approximations in fuzzy-rough systems coincide with fuzzification in logical-type fuzzy systems. Therefore, the proposed methods can be viewed as extensions to the conventional non-singleton fuzzification method. Fuzzy-rough sets in the sense of Nakamura result with a formulation of a general fuzzy-valued fuzzy logic system. For this purpose, three realizations of general fuzzy-valued fuzzy systems: triangular, trapezoidal and Gaussian, are presented in details.

4.1 State of the Art

As it is depicted in Fig. 4.1 the classical fuzzy logic system consists of a base of fuzzy rules, an inference engine returning a fuzzy conclusion on the basis
of a fuzzy premise and fuzzy rules, a defuzzifier reducing fuzzy conclusion set to a crisp value, and an optional input fuzzifier converting crisp input values into a fuzzy premise set (see e.g. [Rutkowski 2004b, 2008]).

The knowledge base for a typical MISO (multiple input single output) system contains \( k \) pairs of fuzzy antecedents and fuzzy consequents forming the following rules

\[
R_k : \text{IF } x \text{ is } A_k \text{ THEN } y \text{ is } B_k, \quad (4.1)
\]

where \( x \) is an \( N \)-dimensional premise and \( A_k \) is an \( N \)-dimensional antecedent fuzzy set, and \( B_k \) is a scalar consequent fuzzy set used in the \( k \)-th rule, \( k = 1, \ldots, K \).

There are two approaches to interpret the conditional statement IF-THEN. One common in fuzzy control employs t-norms, hence, a rule function \( R \) is a conjunction

\[
R(x, y) = T (x_1, \ldots, x_N, y). \quad (4.2)
\]

The second approach employs material implications (see Rutkowska et al 2002; Rutkowski 2004a; Rutkowski and Cplaka 2003 for more detailed study), for which the following constructions can be enumerated:

- (strong) s-implications

\[
R(x, y) = S \left( N(x), y \right), \quad (4.3)
\]

- (residual) r-implications

\[
R(x, y) = \sup_{z \in [0,1]} \{ z | T(x, z) \leq y \}, \quad (4.4)
\]

- (quantum logic) ql-implications

\[
R(x, y) = S \left( N(x), T(x, y) \right), \quad (4.5)
\]

- (Dishkant) d-implications

\[
R(x, y) = S \left( T \left( N(x), N(y) \right), y \right). \quad (4.6)
\]

Exemplary constructions of these implications are presented in Chapt. [1]