Chapter 2  A Survey of Chaos Theory

Abstract  This chapter briefly summarizes chaos theory. The chapter begins with describing chaos as bounded aperiodic random-like deterministic motion, which is sensitive to initial states and thus unpredictable after a certain time of a system. The geometrical structure of chaos is analyzed via the Poincaré map. Three typical routes to chaos are introduced as period-doubling sequence, intermittency, and quasiperiodic torus breakdown. The chapter covers two main numerical approaches to identify chaos, Lyapunov exponents and power spectra. The Melnikov theory is presented to predict the transversal intersection of stable and unstable manifolds of a saddle point. Such an intersection results in complicated dynamical behaviors which are sensitive to initial conditions. Finally, chaos is treated in the context of Hamiltonian systems. KAM theorem is stated without the proof. Two mechanisms of Hamiltonian chaos are illustrated as KAM tori breakup and Arnol’d diffusion. The Melnikov theory is generalized to higher-dimensional Hamiltonian systems.

Keywords  chaos, Poincaré map, period-doubling sequence, intermittency, quasiperiodic torus breakdown, Lyapunov exponents, power spectra, Melnikov theory, KAM theorem, KAM tori breakup, Arnol’d diffusion

This chapter briefly summarizes chaos theory, most of which will be applied in the subsequent chapters. The chapter begins with describing chaos as bounded aperiodic random-like deterministic motion, which is sensitive to initial states and thus unpredictable after a certain time of a system. The geometrical structure of chaos is analyzed via the state space as well as the Poincaré map. Three typical routes to chaos are introduced as period-doubling sequence, intermittency, and quasiperiodic torus breakdown. The chapter covers two main numerical approaches to identify chaos, Lyapunov exponents and power spectra. The Melnikov theory is presented to predict the transversal intersection of stable and unstable manifolds of a saddle point. Such an intersection is explained to result in complicated dynamical behaviors which are sensitive to initial conditions. Finally, chaos is treated in the context of Hamiltonian systems. KAM theorem is stated without the proof. Two mechanisms of Hamiltonian chaos are illustrated as KAM tori breakup and Arnol’d diffusion. The Melnikov theory is generalized to higher-dimensional Hamiltonian systems. This chapter is only a brief survey of chaos, and references [1-6] present
a more comprehensive treatment of chaos with the emphasis on engineering applications.

2.1 The Overview of Chaos

2.1.1 Descriptions of Chaos

Motions of many natural or engineering systems, including attitude motion of spacecraft, can be governed by a set of equations derived from the natural laws such as Newton’s laws or Euler’s equation. The set of equations, defined mathematically as a dynamical system, yields the time evolution of the state of a system from the knowledge of its previous history. Therefore, the state at any time can be determined by the governing equations and the initial states. The equations describing a dynamical system may be algebraic or differential equations.

In modern science, chaos is a term to describe a type of motion, or time evolution resulting from a dynamical system, that appears, on detailed examination, to be completely disordered and extremely complex. The disorder and complicacy are due to the following reasons.

Chaos is a recurrent aperiodic motion. Hence, chaos can be practically defined as a bounded steady-state response that is not an equilibrium state or a periodic motion, or a quasiperiodic motion. For systems with finite degrees of freedom, a bounded response of linear systems must be an equilibrium state, a periodic motion, or a quasiperiodic motion. Hence chaos is a striking feature of a nonlinear system. As a recurrent motion, chaos is bounded so that it will trend to the infinite.

Chaotic motions are also characterized by sensitivity to initial states; that is, tiny differences in the initial conditions can be quickly amplified to produce huge differences in the response. Due to such sensitivity, the long-term prediction for chaos is impossible, because all initial conditions have to be prescribed in a certain precision, while, after enough time, the motion depends on the digits in the conditions beyond the precision. That is, chaos is unpredictable after enough time because a small difference in the initial conditions beyond their precision will result in rapidly (usually exponentially) growing perturbation of the motion. This phenomenon is vividly called butterfly effect. A disturbance caused by the wings of a butterfly in Shanghai can lead to a rainstorm a few days later in Toronto.

Chaos, as a recurrent aperiodic motion, has no pattern or order to follow, just like a stochastic process. Actually, the spectrum of a chaotic motion has a continuous broadband, which is the same as a true random signal. In contrast, the spectra of periodic or quasiperiodic motions consist of a number of sharp spikes. In addition