Domain Decomposition Preconditioning for High Order Hybrid Discontinuous Galerkin Methods on Tetrahedral Meshes

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Abstract. Hybrid discontinuous Galerkin methods are popular discretization methods in applications from fluid dynamics and many others. Often large scale linear systems arising from elliptic operators have to be solved. We show that standard \( p \)-version domain decomposition techniques can be applied, but we have to develop new technical tools to prove poly-logarithmic condition number estimates, in particular on tetrahedral meshes.

1 Introduction

In this paper we are concerned with discontinuous Galerkin (DG) finite element methods for elliptic problems [4, 12, 24]. The motivation might be to have dominant convection, or one wants to build exactly divergence free finite element spaces for incompressible flows [11, 31], or other. We think of operator splitting methods, where one has to solve a large scale symmetric matrix equation in each time-step.

In recent years hybridization methods appeared, which allow to reduce the discrete system to the element interfaces [10]. This paper is concerned with the construction and analysis of domain decomposition methods for the Hybrid Discontinuous Galerkin (HDG) method. We consider one element as sub-domain, and the coarse grid problem consists of mean values on element interfaces. We prove
robustness with respect to the mesh-size, and a poly-logarithmic growth of the condition number with the polynomial order \( p \).

There is now an established literature on high order finite element methods, from the more theoretical point of view as well as from an applied one \([13, 26, 41, 43]\).

We consider two strategies for domain decomposition algorithms \([45]\), non-overlapping Schwarz type methods \([15, 20, 21]\) and balancing domain decomposition with constraints (BDDC) \([14, 33]\). There is a big literature, in particular high order methods and three dimensional problems are treated in \([2, 5, 7, 8, 9, 19, 23, 27, 28, 30, 31, 36, 37, 38, 40, 42]\). There is a classical paper on multi-level analysis for h-version DG methods by Gopalakrishnan and Kanschat \([18]\), and a recent one studying higher order methods by Antonietti and Houston \([3]\) showing a polynomial growth of the condition number in \( p \). We will see that the conditioning is significantly improved by hybridization, namely to a poly-logarithmic growth. We are not aware of particular analysis for preconditioners for high order HDG methods, even not in 2D.

The main result of the present paper is Theorem 3 proving that optimal extension from faces to elements with Dirichlet constraints is nearly as good as extension without constraints. With this result condition number estimates follow with the usual techniques.

The main difficulty is to build optimal extension operators from an edge to a tetrahedron. This problem was solved for hexahedral elements by multiplying with fast decaying functions by Pavarino and Widlund \([38]\). Polynomial extension operators for simplicial elements are usually based on smoothing operators \([5, 34]\). Heuer and Leydecker have analyzed such operators also for boundary elements, i.e., for three dimensional edge to face extension.

We cannot use the existing simplicial extension operators to prove quasi-optimality of HDG methods since they do not decay fast enough in the jump-norm. We give a new construction of discrete edge-to-tetrahedron extension operators which are motivated by the multiplication with low-energy functions of Pavarino and Widlund, but are contained in the polynomial space on tetrahedra.

We declare some notation. With \( a \lesssim b \) we mean the existence of a generic constant \( c \) such that \( a \leq cb \), where \( c \) is independent of parameters \( h \) and \( p \). Otherwise, we denote the dependence as \( c(p) \). The space of univariate polynomials of order \( p \) is \( P^p \), and \( P^p(T) \) is the space of multivariate polynomials of total order \( p \) on a simplex \( T \). To simplify notation we redefine \( \log p := 1 \) for \( p \in \{0, 1\} \).

In Sect. 2 we give the hybrid DG formulation, in Sect. 3 we prove the main result, Theorem 3, and show how to apply it to analyze domain decomposition algorithms for HDG. Technical lemmas are shifted to Sect. 4, 5, and 6. In Sect. 4 we collect properties of orthogonal polynomials, and prove one dimensional trace estimates and construct one-dimensional extension operators with respect to different norms. The short Sect. 5 gives the proofs for extension from vertices, the technical proofs for the extension from edges are in Sect. 6.