Characterizing Certain Topological Specifications*

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Abstract. We prove a characterization theorem à la van Benthem for a particular modal system called topologic, which is, among other things, suitable for specifying the interrelation between knowledge and topology. The comparison language arising naturally from the relevant semantics is well-known from the beginnings of topological model theory, and subset space bisimulations provide for the proper notion of invariance of formulas here.

1 Introduction

As is known, modal languages constitute a widespread formalism for the purposes of system specification, e.g., in the case of distributed or multi-agent scenarios. But once a specification language is to hand, trying to get to the bottom of its scope is a fundamental issue ever. The classical van Benthem Characterization Theorem gives a satisfactory answer to this problem in the case of basic modal logic: the bisimulation-invariant fragment of first-order logic and the entirety of those properties that can be specified by modal formulas coincide; see [4], Theorem 2.68. In this paper, we prove an analogous result for Moss and Parikh’s bimodal language for topological spaces, $\mathcal{L}$, originating from [12] (see also [7]).

Let us take a quick look at the semantics of $\mathcal{L}$. Its basic units are composed of two ingredients, to wit, the actual state $x$ of the world and an open neighborhood $U$ of $x$. In knowledge-theoretic contexts, which serve us as an example here, $U$ may be viewed as the current epistemic state of an agent under discussion, i.e., the set of those states that cannot be distinguished by what the agent topically knows. The two modalities of the language, $K$ and $\Box$, quantify across all elements of $U$ and ‘downward’ over all open sets contained in $U$, respectively. Thus $K$ captures the common notion of knowledge (see [8]), and $\Box$ reflects effort to acquire knowledge since gaining knowledge goes hand in hand with a shrinkage of the epistemic state. In fact, knowledge acquisition is reminiscent of a topological procedure in this way. The appropriate logic for topological spaces, exactly topologic, was first determined by Georgatos in his thesis [10]. Meanwhile, a considerable amount of work was involved in the development of a corresponding

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theory; see Ch. 6 of the handbook [2] for a guide to the literature up to 2006. Summing all this and the more recent achievements up, one may state that the emerging system embodies a well-established toolkit for specification tasks related to reasoning about knowledge as well as topological reasoning, with the two-valued semantics involving both states and ambient open sets making up its distinctive feature.

Now is the time to explore the expressive power of the language \( \mathcal{L} \). A careful analysis of its semantics will direct us to our goal, viz a preferably comprehensive description of \( \mathcal{L} \)-definable properties. To this end, we shall first identify the right comparison language for \( \mathcal{L} \). It shows that this language coincides with the language \( \mathcal{L}_2 \) for so-called weak structures, introduced by Flum and Ziegler in the monograph [9]. See that \( \mathcal{L}_2 \) is a first-order language in essence, the machinery of first-order model theory (see, e.g., [6]) is applicable to our problem, and we shall make an extensive use of this in due course.

The very topic of Flum and Ziegler’s book is a sublanguage of \( \mathcal{L}_2 \) where quantification over set variables is restricted in a certain manner. This language, \( \mathcal{L}_t \), has proven a perfect match for the classic topological interpretation of usual (mono-)modal logic (see [1]), as the recent paper [5] shows. Unfortunately, we cannot exploit the expressivity issues obtained in that paper for our purposes, since the translations of \( \mathcal{L} \)-formulas do not generally satisfy those syntactic restrictions. The prize we have to pay for obtaining the desired characterization by means of the contemplated methods nevertheless, is to admit arbitrary subset spaces as semantic structures, as it is the case with the original Moss-Parikh logic.

The rest of this paper is organized as follows. In the next section, we supply the basic definitions from [7] needed subsequently, and we define a standard translation of the set of all \( \mathcal{L} \)-formulas which is induced by the semantics of \( \mathcal{L} \). Finally in this section, we reason about the image of that translation. In Section 3, the concept of saturated models is revisited. Moreover, the idea of modal saturation that is relevant to our setting is investigated. Section 4 is devoted to bisimulations, in fact, subset space bisimulations introduced in [3]. The desired characterization theorem is then proved in Section 5. At the end of the paper, we add some concluding remarks.

2 The Languages We Consider

In this section, we first fix the language \( \mathcal{L} \) underlying topologic and extract a translation into first-order logic from that. Afterwards, we recall the language \( \mathcal{L}_2 \) before connecting \( \mathcal{L} \) to it.

To begin with, we define the syntax of \( \mathcal{L} \). Let \( \text{Prop} = \{ p, q, \ldots \} \) be a denumerably infinite set of symbols called proposition variables (which should represent the basic facts about the states of the world). Then, the set \( \text{Form} \) of all \( \mathcal{L} \)-formulas over \( \text{Prop} \) is defined by the rule \( \alpha ::= \top | p | \neg \alpha | \alpha \land \alpha | K \alpha | \Box \alpha \). Later on, the boolean connectives that are missing here are treated as abbreviations, as needed. The dual operators of \( K \) and \( \Box \) are denoted by \( L \) and \( \Diamond \), respectively; \( K \) is called the knowledge operator and \( \Box \) the effort operator.