Chapter 11
Poisson–Lie Groups

A Poisson–Lie group \((G, \pi)\) is a Lie group \(G\) which is equipped with a Poisson structure \(\pi\). The Poisson structure is demanded to be compatible with the group structure, in the sense that one requires the product map \(G \times G \to G\) to be a morphism of Poisson manifolds, where \(G \times G\) is equipped with the product Poisson structure.

To a Lie group \(G\) there corresponds a Lie algebra \(\mathfrak{g}\), which is the tangent space to \(G\) at \(e\), where the Lie bracket \([\cdot, \cdot]_\mathfrak{g}\) on \(\mathfrak{g}\) is inherited from the Lie bracket on the space of (left-invariant) vector fields on \(G\). If, in addition, \(G\) comes equipped with a Poisson structure, the Lie algebra \(\mathfrak{g}\) inherits the structure of a Lie coalgebra, which amounts to a Lie algebra structure \([\cdot, \cdot]_\mathfrak{g}^*\) on the dual vector space \(\mathfrak{g}^*\). In the case of a Poisson–Lie group \((G, \pi)\), the fact that the multiplication in \(G\) and the Poisson structure on \(G\) are compatible, implies a compatibility between the two Lie brackets \([\cdot, \cdot]_\mathfrak{g}\) and \([\cdot, \cdot]_\mathfrak{g}^*\): the transpose to \([\cdot, \cdot]_\mathfrak{g}^*\) is a derivation of the Lie bracket \([\cdot, \cdot]_\mathfrak{g}\).

Formalizing this property leads to the notion of a Lie bialgebra. Thus, to every Poisson–Lie group, there corresponds a Lie bialgebra. Moreover, to every homomorphism of Poisson–Lie groups there naturally corresponds a homomorphism of Lie bialgebras, hence the correspondence between Poisson–Lie groups and Lie bialgebras is functorial. Most importantly, the converse also holds. As we know from Lie’s third theorem, every finite-dimensional Lie algebra is the Lie algebra of some Lie group, which can be chosen to be connected and simply connected. This Lie group can be made into a Poisson–Lie group if the Lie algebra is a Lie bialgebra.

Poisson–Lie groups are introduced in Section 11.1, while their infinitesimal analogs, Lie bialgebras, are introduced in Section 11.2. The above correspondence between Lie bialgebras and Poisson–Lie groups is discussed in Section 11.3. Using dressing actions, we study the symplectic leaves of Poisson–Lie groups in Section 11.4.

All Lie groups in this chapter are real (\(\mathbb{F} = \mathbb{R}\)) or complex (\(\mathbb{F} = \mathbb{C}\)).
11.1 Multiplicative Poisson Structures and Poisson–Lie Groups

A Poisson structure on a Lie group $G$ makes $G$ into a Poisson manifold. For reasons which will be given later in this chapter, one demands the following compatibility relation between the Poisson structure on $G$ and the group structure of $G$.

**Definition 11.1.** A Poisson structure $\pi$ on a (real or complex) Lie group $G$ is said to be *multiplicative* if the product map $\mu : G \times G \to G$ is a Poisson map, where $G \times G$ is endowed with the product Poisson structure. The pair $(G, \pi)$ is then called a *Poisson–Lie group*.

A map $\Phi : G \to H$ between two Poisson–Lie groups $(G, \pi)$ and $(H, \pi')$ is called a *Poisson–Lie group homomorphism* if it is both a Poisson map and a group homomorphism.

**11.1.1 The Condition of Multiplicativity**

We give in the following proposition two useful characterizations of the multiplicativity of a Poisson structure on a Lie group. Recall that for $g \in G$ the maps of left and right translation $L_g$ and $R_g$ in $G$ are defined by $L_g(h) := gh$ and $R_g(h) := hg$, for all $h \in G$.

**Proposition 11.2.** Let $G$ be a Lie group. For a Poisson structure $\pi$ on $G$, the following three conditions are equivalent:

(i) $\pi$ is multiplicative;

(ii) For all $g, h \in G$:

\[ \pi_{gh} = \wedge^2(T_g R_h) \pi_g + \wedge^2(T_h L_g) \pi_h ; \]  

(11.1)

(iii) The map $\Psi : G \to \wedge^2 g$, which is defined for all $g \in G$ by

\[ \Psi(g) := \wedge^2(T_g R_{g^{-1}}) \pi_g , \]  

(11.2)

is a cocycle of $G$, with respect to the adjoint representation of $G$ on $\wedge^2 g$, i.e.,

\[ \Psi(gh) = \Psi(g) + \text{Ad}_g \Psi(h) , \]  

(11.3)

for all $g, h \in G$.

**Proof.** We first prove that (i) and (ii) are equivalent, which amounts to proving that $\mu : G \times G \to G$ is a Poisson map if and only if (11.1) holds for all $g, h \in G$. Let us denote the product Poisson structure on $G \times G$ by $\Pi$; according to (2.11), the value of $\Pi$ at $(g, h) \in G \times G$ is given by the bivector

\[ \Pi_{(g, h)} = \wedge^2(T_g t_h) \pi_g + \wedge^2(T_h t_g') \pi_h , \]  

(11.4)