Chapter 7
Linear Poisson Structures and Lie Algebras

Together with symplectic manifolds, considered in the previous chapter, Lie algebras provide the first examples of Poisson manifolds. Namely, the dual $\mathfrak{g}^*$ of a finite-dimensional Lie algebra $\mathfrak{g}$ admits a natural Poisson structure, called its Lie–Poisson structure, which provides new insights and technical tools in the study of Lie algebras. For example, the coadjoint orbits in $\mathfrak{g}^*$ are precisely the symplectic leaves of the Lie–Poisson structure, showing in particular that coadjoint orbits are even-dimensional.

Lie–Poisson structures are linear Poisson structures, in the sense that they are Poisson structures on a vector space V, for which the Poisson bracket of every pair of *linear* functions on V (elements of V^*) is a *linear* function on V. Also, every linear Poisson structure (on a finite-dimensional vector space) is a Lie–Poisson structure and we have a functorial equivalence between linear Poisson structures and Lie algebra structures.

In many cases, one considers the Poisson structure on the Lie algebra $\mathfrak{g}$ itself, rather than on $\mathfrak{g}^*$. This is usually done by identifying $\mathfrak{g}$ with $\mathfrak{g}^*$, using a non-degenerate Ad-invariant symmetric bilinear form on $\mathfrak{g}$, which exists for example for every semisimple Lie algebra. The coadjoint orbits are then identified with the adjoint orbits and the Hamiltonian vector fields on $\mathfrak{g}$ take a natural form, a so-called Lax form.

The Lie–Poisson structure on $\mathfrak{g}^*$ is introduced in Section 7.1, while the induced Poisson structure on $\mathfrak{g}$ is discussed in Section 7.2. The main properties of the Lie–Poisson structure are given in Section 7.3. A variant of Lie–Poisson structures, namely affine (= linear + constant) Poisson structures and their Lie theoretical interpretation is discussed in Section 7.4. We finish this chapter with a short introduction to the linearization of Poisson structures (in the neighborhood of a point where the rank is zero).

Unless otherwise stated, $\mathbb{F}$ denotes an arbitrary field of characteristic zero.

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7.1 The Lie–Poisson Structure on $\mathfrak{g}^*$

Suppose that $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbb{F}$, with Lie bracket $[\cdot\,,\cdot]$. We denote the dual vector space to $\mathfrak{g}$ by $\mathfrak{g}^*$. We associate to each element[1] e of $\mathfrak{g}$, a linear function $e^* : \mathfrak{g}^* \to \mathbb{F}$, defined by

$$\begin{aligned} e^* : \mathfrak{g}^* &\to \quad \mathbb{F} \\ \xi &\mapsto \langle \xi, e\rangle := \xi(e)\,. \end{aligned}$$

In words, e^* is "evaluation at e". Taking a basis $(e_1,\dots,e_d)$ of $\mathfrak{g}$, we obtain a system of linear coordinates $(x_1,\dots,x_d)$ on $\mathfrak{g}^*$ by letting $x_i := e_i^*$. We use it to build the skew-symmetric matrix $X = (x_{ij})_{1\leqslant i,j\leqslant d}$, whose elements are defined by $x_{ij} := [e_i,e_j]^*$; each entry of X is a function on $\mathfrak{g}$, which can be expressed as a linear combination of the coordinates $x_1,\dots,x_d$. Consider the skew-symmetric biderivation $\{\cdot\,,\cdot\}$ of $\mathbb{F}[x_1,\dots,x_d]$, defined by setting, for $F,G \in \mathbb{F}[x_1,\dots,x_d]$,

$$\{F,G\} := \sum_{i,j=1}^{d} x_{ij}\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j}\,. \tag{7.1}$$

It is the unique skew-symmetric biderivation of $\mathbb{F}[x_1,\dots,x_d]$ such that $\left\{e_i^*,e_j^*\right\} = [e_i,e_j]^*$, for all $1 \leqslant i < j \leqslant d$. By bilinearity of $\{\cdot\,,\cdot\}$ and $[\cdot\,,\cdot]$,

$$\{e^*,f^*\} = [e,f]^*\,, \tag{7.2}$$

for all $e,f \in \mathfrak{g}$. On the one hand, this implies that Definition (7.1) is independent of the chosen basis $(e_1,\dots,e_d)$. On the other hand, it yields, for all i,j,k with $1 \leqslant i,j,k \leqslant d$,

$$\left\{\left\{x_i,x_j\right\},x_k\right\} = [[e_i,e_j],e_k]^*\,,$$

so that the Jacobi identity for $[\cdot\,,\cdot]$ implies that $\left\{\left\{x_i,x_j\right\},x_k\right\} + \circlearrowleft (i,j,k) = 0$ for all $1 \leqslant i < j < k \leqslant d$. According to Proposition 1.8, this shows that $\{\cdot\,,\cdot\}$ is a Poisson structure on $\mathbb{F}[x_1,\dots,x_d]$. Similarly, according to *(iii)* in Proposition 1.36, Eq. (7.1) also defines a Poisson structure on the algebra of smooth functions on $\mathfrak{g}^*$, when $\mathbb{F} = \mathbb{R}$, or on the algebra of holomorphic functions on $\mathfrak{g}^*$, when $\mathbb{F} = \mathbb{C}$; in either case, it is the unique Poisson structure which satisfies (7.2). Thus, $\mathscr{F}(\mathfrak{g}^*)$ inherits a Poisson structure from the Lie bracket on $\mathfrak{g}$, irrespective of whether we take $\mathscr{F}(\mathfrak{g}^*)$ to be the algebra of polynomial, smooth, or holomorphic functions on $\mathfrak{g}^*$.

An intrinsic formula for $\{F,G\}$ can be written down in terms of the differentials of F and G. Since the differential of $F \in \mathscr{F}(\mathfrak{g}^*)$ at ξ is a linear map $\mathrm{d}_\xi F : T_\xi\mathfrak{g}^* \to \mathbb{F}$, and since $T_\xi\mathfrak{g}^*$ is naturally isomorphic with $\mathfrak{g}^*$, we can think of $\mathrm{d}_\xi F$ as being an element of $(\mathfrak{g}^*)^*$, i.e., as an element of $\mathfrak{g}$, since $\mathfrak{g}$ and its bidual are canonically isomorphic (recall that $\mathfrak{g}$ is finite-dimensional; the canonical isomorphism $\mathfrak{g} \to (\mathfrak{g}^*)^*$

[1] We use in this section letters e and f to denote elements of a Lie algebra $\mathfrak{g}$, to reserve the letters x,y,z for elements of the bidual $(\mathfrak{g}^*)^*$. After identification of $\mathfrak{g}$ with its bidual, the letters x,y,z become elements of $\mathfrak{g}$, just like everywhere else in the book.