Low-dimensional Poisson manifolds are often used as toy models, to obtain a better understanding of the theory of Poisson manifolds, as well as to illustrate their unexpected complexity. In the two-dimensional case, for example, describing all Poisson structures on the affine space $\mathbb{R}^2$ is deceivingly simple, because the Jacobi identity is satisfied for all bivector fields; however, their local classification is non-trivial, and has up to now only been accomplished under quite strong regularity assumptions on the singular locus of the Poisson structure, which can be identified with the zero locus of a local function on the manifold. As a result, the study of Poisson structures in two dimensions already takes us to the non-trivial singularity theory of functions of two variables!

Dimension three is the smallest dimension in which the Jacobi identity for a bivector field is not always satisfied. In this dimension, the Jacobi identity can be stated as an integrability condition of a distribution, which eventually leads to the symplectic foliation, or as the integrability condition of a differential one-form, dual to the Poisson structure with respect to a volume form (assuming that the manifold is orientable). Despite the extra complexity which comes from the extra dimension, Poisson structures in dimension three have one key property in common with Poisson structures in dimension two: their rank is equal to two (except when the Poisson structure is trivial). Many results about three-dimensional Poisson manifolds are essentially true because the Poisson structure is of rank two. A key example of such a result is the characterization of a regular Poisson structure of rank two in terms of its symplectic foliation.

Poisson structures in dimensions two and three are presented in different sections (Sections 9.1 and 9.2). In both cases, we pay special attention to the global/local and geometrical/algebraic aspects of these Poisson structures and we prove at the end of each section a (partial) classification result.

Unless otherwise stated, $\mathbb{F}$ is an arbitrary field of characteristic zero.
9.1 Poisson Structures in Dimension Two

In this section, we study Poisson structures on manifolds or affine varieties of dimension two. Since a Poisson structure is a bivector field or a biderivation, the smallest dimension which one can consider for studying non-trivial Poisson structures is dimension two. In this case, the Jacobi identity is always satisfied, so that every bivector field or skew-symmetric biderivation is a Poisson structure. We present Poisson structures in dimension two first from the global point of view, then from the local point of view and finally we discuss their formal classification.

9.1.1 Global Point of View

First of all, let $M$ be a manifold (real or complex, depending on whether $F = \mathbb{R}$ or $F = \mathbb{C}$) of dimension two and let $\mathcal{F}(M)$ be the algebra of smooth or holomorphic functions on $M$. According to Proposition 3.5, every bivector field $\pi$ on $M$ is a Poisson structure because $[\pi, \pi]_S$ is a trivector field on $M$, hence is zero. We conclude that the space of all Poisson structures on $M$ coincides with the vector space $\mathcal{X}^2(M)$ of all bivector fields on $M$. In particular, two Poisson structures on $M$ are always compatible (see Section 3.3.2). For every point $m$ of $M$, the rank of $\pi$ at $m$ is zero or two. Discarding the case of the trivial Poisson structure, it follows that the singular locus of $(M, \pi)$ coincides with the set of points of $M$ where $\pi$ vanishes.

Next, let us consider a complex affine surface $M \subset \mathbb{C}^d$ of dimension two, equipped with its algebra $\mathcal{F}(M) = \mathbb{C}[x_1, \ldots, x_d]/\mathcal{I}$ of regular functions. We show that every skew-symmetric biderivation $\pi$ of $\mathcal{F}(M)$ is a Poisson structure on $M$. To do this, we prove that every skew-symmetric triderivation of $\mathcal{F}(M)$ is zero; since $[\pi, \pi]_S$ is a skew-symmetric triderivation of $\mathcal{F}(M)$, it follows that $\pi$ is a Poisson structure on $M$. Let us denote by $M^{\text{sm}}$ the set of all smooth points of $M$. Let $P$ be a skew-symmetric triderivation on $M$ and suppose that $P$ is different from zero at some point $m \in M$. Since $P$ can be written as

$$P = \sum_{1 \leq i < j < k \leq d} P_{ijk} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k},$$

where $P_{ijk} := P[x_i, x_j, x_k]$, this means that $P_{ijk}(m) \neq 0$, for some $i, j, k$. It follows that $P$ is different from zero in a neighborhood of $m$ in $M$. Since $M^{\text{sm}}$ is dense in $M$, this means that $P$ is non-zero in the neighborhood of some smooth point of $M$. However, as we have seen in Section 2.3.2, the set of smooth points of $M$ carries a natural structure of a complex manifold, and every skew-symmetric triderivation of $\mathcal{F}(M)$ corresponds on it to a trivector field. Since $M^{\text{sm}}$ is a two-dimensional manifold, every trivector field on $M^{\text{sm}}$ is zero, hence $P$ is zero on $M^{\text{sm}}$, and we arrive at a contradiction. This shows that $P$ vanishes at every point of $M$, hence that every skew-symmetric biderivation $\pi$ of $\mathcal{F}(M)$ is a Poisson structure on $M$. As in the case of two-dimensional Poisson manifolds, the rank of $\pi$ can only take the