Preventing Unraveling in Social Networks: The Anchored k-Core Problem

Kshipra Bhawalkar[1], Jon Kleinberg[2], Kevin Lewi[1], Tim Roughgarden[1], and Aneesh Sharma[3]

[1] Stanford University, Stanford, CA, USA
[2] Cornell University, Ithaca, NY, USA
[3] Twitter, Inc.

Abstract. We consider a model of user engagement in social networks, where each player incurs a cost to remain engaged but derives a benefit proportional to the number of engaged neighbors. The natural equilibrium of this model corresponds to the *k-core* of the social network — the maximal induced subgraph with minimum degree at least k.

We study the problem of "anchoring" a small number of vertices to maximize the size of the corresponding anchored k-core — the maximal induced subgraph in which every non-anchored vertex has degree at least k. This problem corresponds to preventing "unraveling" — a cascade of iterated withdrawals. We provide polynomial-time algorithms for general graphs with $k = 2$, and for bounded-treewidth graphs with arbitrary k. We prove strong inapproximability results for general graphs and $k \geq 3$.

1 Introduction

A defining property of social networks — where nodes represent individuals, and edges represent friendships — is that the behavior of an individual is influenced by that of his or her friends. In particular, they often exhibit positive "network effects", where the utility of an individual is increasing in the number of friends that behave in a certain way. For example, empirical work has determined that individuals are more likely to contribute useful content to a social network if their friends do [4]. Increasingly, empirical studies suggest that the influence of interactions in social network extends to behavior outside of these networks, as well [9]. An obvious question, studied from a system-building perspective in [10], is how to design or modify social networks to maximize the participation and engagement of its users.

For concreteness, consider scenarios where each individual of a social network has two strategies, to "engage" or to "drop out". Being engaged could mean contributing to a public good (like network content), signing up for a new social network feature, adopting one technology instead of another, and so on. We assume that a player is more likely to be engaged if many friends are. For this Introduction, we focus on our most basic model. We first describe our model via

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a process of cascading withdrawals, and then formulate it using a simultaneous-move game.

Assume that all individuals are initially engaged, and for a parameter k, a node remains engaged if and only if at least k friends are engaged. For example, engagement could represent active participation in the social network, which is worthwhile to an individual if and only if at least k friends are also actively participating. Or, dropping out could represent the abandonment of an incumbent product in favor of a newly arrived competitor; when the number of one's friends using the old product falls below k, one switches to the new product.

In this basic model, it is clear that all individuals with less than k friends will drop out. These initial withdrawals can be contagious, spreading to individuals with many more than k friends. See Figure 1 for an example of this phenomenon. In general, when such iterated withdrawals die out, the remaining engaged individuals correspond to a well-known concept in graph theory — the *k-core* of the original social network, which by definition is the (unique) maximal induced subgraph with minimum degree at least k. Alternatively, the k-core is the (unique) result of iteratively deleting nodes that have degree less than k, in any order.

Schelling [17, P.214] describes this type of "unraveling" in typically picturesque language, by contrasting the cycle with the line (with $k = 2$). He imagines people sitting with reading lamps, each of whom can get additional partial illumination from the lamps of their neighbor(s):

> *In some cases the arrangement matters. If everybody needs 100 watts to read by and a neighbor's bulb is equivalent to half one's own, and everybody has a 60-watt bulb, everybody can read as long as he and both his neighbors have their lights on. Arranged in a circle, everybody will keep his light on if everybody else does (and nobody will if his neighbors do not); arranged in a line, the people at the ends cannot read anyway and the whole thing unravels.*

A Game-Theoretic Formulation. The k-core can be seen as the maximal equilibrium in a natural game-theoretic model; it has been studied previously in this guise in the social sciences literature [5,6,16]. Concretely, imagine that each node in a social network G is considering whether to remain engaged in a social activity. We suppose that each node v in G incurs an (integer) cost of $k > 0$ for the effort it must spend to remain engaged. Node v also obtains a benefit of 1 from each neighbor w who is engaged; this reflects the idea that the benefit from participation in the activity comes from interaction with neighbors in the social network.

If each node makes its decision simultaneously, we can model the situation as a simultaneous-move game in which the nodes are the players, and v's possible strategies are to remain engaged or to drop out. For a choice of strategies σ by each player, let S_σ be the set of players who choose to remain engaged. The payoff of v is 0 if it drops out, and otherwise it is v's degree in the induced subgraph $G[S_\sigma]$ minus k. Note that we can talk about sets of engaged nodes and strategy profiles interchangeably.