Crossing Angles of Geometric Graphs

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Abstract. We study the crossing angles of geometric graphs in the plane. We introduce the crossing angle number of a graph $G$, denoted $\text{can}(G)$, which is the minimum number of angles between crossing edges in a straight line drawing of $G$. We show that an $n$-vertex graph $G$ with $\text{can}(G) = O(1)$ has $O(n)$ edges, but there are graphs $G$ with bounded degree and arbitrarily large $\text{can}(G)$. We also initiate studying the global crossing-angle rigidity of geometric graphs. We construct bounded degree graphs $G = (V,E)$ such that for any two straight-line drawings of $G$ with the same prescribed crossing angles, there is a subset $V' \subset V$ of $|V'| \geq |V|/2$ vertices that are similar in the two drawings.

1 Introduction

Graphs with $n$ vertices and more than $3n - 6$ edges are not planar and their drawings in the plane have crossing edges. The crossing number $\text{cr}(G)$ (resp., rectilinear crossing number $\overline{\text{cr}}(G)$) of a graph $G$ is the minimum number of crossings in any drawing (resp., straight-line drawing) of $G$ in the plane. Bienstock and Dean [4] showed that there are families of graphs with unbounded rectilinear crossing number, but crossing number at most $k$, for any $k \geq 4$. However, $\overline{\text{cr}}(G) = \text{cr}(G)$ if $\text{cr}(G) \leq 3$. Moreover, there are families of bounded degree graphs (even cubic graphs [18]) with arbitrarily large crossing numbers.

Angle conditions on the crossing edges have only been recently considered. The motivation comes from cognitive experiments, which show that having small crossing angles is negatively correlated to path-tracking ability in a graph drawing. Previous research focused on the crossing resolution [6] of straight-line (or polyline) drawings, that is, the minimum angle at which crossing edges meet. In this paper, we consider the number of different angles between crossing edges. A crossing angle in a straight-line drawing of a graph is an angle $\alpha$, $0 < \alpha \leq \frac{\pi}{2}$, between two crossing edges. The crossing angle number of a graph $G$, denoted $\text{can}(G)$, is the minimum number of crossing angles in a straight line drawing of $G$. In Section 2, we show that every $n$-vertex graph $G$ has less than $(6 \text{can}(G) + 3)n$ edges. We also show that for every $\varepsilon > 0$, there are $n$-vertex graphs of maximum degree $O(1/\varepsilon)$ such that $\text{can}(G) = \Omega(n^{1/2-\varepsilon})$.

Global Rigidity. A graph $G = (V,E)$ is globally rigid in the plane if for every function $\ell : E \to \mathbb{R}^+$, any two straight-line drawings of $G$ in which the Euclidean
length of each edge \( e \in E \) is \( \ell(e) \) are congruent. In other words, the edge lengths determine at most one straight-line drawing up to congruency. For instance, complete graphs are globally rigid. Saxe [19] showed that it is strongly NP-hard to decide whether a graph is globally rigid. Jackson and Jordan [14] gave a simple combinatorial characterization of generic global rigidity, where the edge lengths determine at most one straight-line drawing (up to congruency) if the vertices are in general position. They also extended this notion to a so-called length-direction rigidity [15], where each edge has either a prescribed length or a prescribed direction vector.

Global Crossing-Angle Rigidity. We adapt the notion of global rigidity to crossing angles. Let \( G = (V, E) \) be a graph with a crossing-angle function \( \alpha : E^2 \to [0, \pi) \cup \{\star\} \). We say that a straight-line drawing of \( G \) complies with \( \alpha \) if for every crossing pair of edges \( (e, f) \), a counterclockwise rotation through \( \alpha(e, f) \) carries the supporting line of \( e \) to that of \( f \); and for every noncrossing pair of edges, we have \( \alpha(e, f) = \star \). In a first approach, we would like to find graphs \( G \) where every function \( \alpha : E^2 \to [0, \pi) \cup \{\star\} \) complies with at most one straight-line drawing up to similarity. This requirement is too strict: we will see that no graph with \( n \geq 3 \) vertices (not even the complete graph \( K_n \)) satisfies this condition. Therefore, we relax this condition, and require \( \alpha \) to determine (up to similarity) the locations of at least a constant fraction of the vertices. A graph \( G = (V, E) \) is globally crossing-angle rigid if for every function \( \alpha : E^2 \to [0, \pi) \cup \{\star\} \), and for any two straight-line drawings of \( G \) complying with \( \alpha \), there is a vertex set \( V'(\alpha) \subset V \) of size \( |V'(\alpha)| \geq |V|/2 \) such that the two drawings of \( V' \) are similar. We prove that \( K_{24} \) is globally crossing-angle rigid, and we also construct an infinite family of globally crossing-angle rigid graphs with maximum degree 47 and diameter \( O(\log n) \) for \( n \geq 24 \) vertices.

Extensions and Open Problems. Our result is a first step towards a possible combinatorial characterization of globally crossing-angle rigid graphs. In our definition of globally crossing-angle rigid graphs, \( \alpha \) determines at least half of the vertices up to similarity, but the choice of the ratio \( \frac{1}{2} \) was arbitrary. For every constant \( c \in (0, 1) \), there are infinitely many graphs \( G = (V, E) \) of maximum degree \( \Delta(c) \) where \( \alpha \) determines at least \( c|V| \) vertices up to similarity. It remains an open problem to find the smallest degree bound \( \Delta(c) \) as a function of \( c \). Our crossing angle function \( \alpha : E^2 \to [0, \pi) \cup \{\star\} \) encodes the directed angle between and ordered pair of edges \( (e, f) \in E^2 \). It would be natural to consider an undirected angle crossing function \( \beta : (E^2) \to (0, \pi/2) \cup \{\star\} \) for unordered pairs \( \{e, f\} \in (E^2) \). Our methods can easily be extended to handle this variant of the problem, albeit with higher vertex degrees.

Related Work. Didimo et al. [8] consider graphs that admit straight line drawings where crossing edges meet at a right angle. Such drawings are called RAC (right angle crossing) drawings. They prove that graphs with \( n \) vertices admitting a RAC drawing have at most \( 4n - 10 \) edges, and this bound is best possible. Argyriou [1] showed that it is NP-hard to decide whether a graph admits a RAC