Polynomial Regression on Riemannian Manifolds

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Abstract. In this paper we develop the theory of parametric polynomial regression in Riemannian manifolds. The theory enables parametric analysis in a wide range of applications, including rigid and non-rigid kinematics as well as shape change of organs due to growth and aging. We show application of Riemannian polynomial regression to shape analysis in Kendall shape space. Results are presented, showing the power of polynomial regression on the classic rat skull growth data of Bookstein and the analysis of the shape changes associated with aging of the corpus callosum from the OASIS Alzheimer’s study.

1 Introduction

The study of the relationship between measured data and known descriptive variables is known as the field of regression analysis. As with most statistical techniques, regression analyses can be broadly divided into two classes: parametric and non-parametric. The most widely known parametric regression methods are linear and polynomial regression in Euclidean space, wherein a linear or polynomial function is fit in a least-squares fashion to observed data. Such methods are the staple of modern data analysis. The most common non-parametric regression approaches are kernel-based methods and spline smoothing approaches which provide much more flexibility in the class of regression functions. However, their non-parametric nature presents a challenge to inference problems; if, for example, one wishes to perform a hypothesis test to determine whether the trend for one group of data is significantly different from that of another group.

Fundamental to the analysis of anatomical imaging data within the framework of computational anatomy is the analysis of transformations and shape which are best represented as elements of Riemannian manifolds. Classical regression suffers from the limitation that the data must lie in a Euclidean vector space. When data are known to lie in a Riemannian manifold, one approach is to map the manifold into a Euclidean space and perform conventional Euclidean analysis. Such extrinsic or coordinate-based methods inevitably lead to complications. For example, given simple angular data, averaging in the Euclidean setting leads to nonsensical results which depend on the chosen coordinates; in the usual coordinates, the mean of 359 degrees and 1 degree is 180 degrees. In the case of shape analysis, regression against landmark data in a Euclidean setting, even after Procrustes alignment (an intrinsic manifold-based technique), will introduce scale and rotation in the regression function, and hence does not capture
invariant properties of shape. By performing an intrinsic regression analysis on
the manifold of shapes, such factors are guaranteed to not effect the analysis.

In previous work, non-parametric kernel-based and spline-based methods have
been extended to observations that lie on a Riemannian manifold with some
success [23], but intrinsic parametric regression on Riemannian manifolds has
received limited attention. Most recently, Fletcher [4] and Niethammer et al.
[5] have each independently developed geodesic regression which generalizes the
notion of linear regression to Riemannian manifolds.

The goal of the current work is to extend the geodesic model in order to
accommodate more flexibility while remaining in the parametric setting. The
increased flexibility introduced by the methods in this manuscript allow a better
description of the variability in the data, and ultimately will allow more powerful
statistical inference. Our work builds off that of Jupp & Kent [3], whose method
for fitting parametric curves to the sphere involved intricate unwrapping and
rolling processes. The work presented in this paper allows one to fit polynomial
regression curves on a general Riemannian manifold, using intrinsic methods and
avoiding the need for unwrapping and rolling. Since our model includes time-
reparametrized geodesics as a special case, information about time dependence
may be obtained from the regression without explicit modeling by examining
the collinearity of the estimated parameters.

The generality of the Riemannian framework developed in this work allows
this parametric regression technique to be applied in a wide variety of contexts
besides shape analysis. Rigid and non-rigid motion can be modeled with polyno-
mials in the Lie groups SO(3), SE(3), or Diff(Ω) [6,7]. Alternatively, the action
of a Riemannian Lie group on a homogeneous space is often a convenient set-
ting, such as in the currents framework [8], or the action of diffeomorphisms
on images [9]. Additionally, there are applications in which the data are best
modeled directly in a Riemannian manifold, such as modeling directional data
on the sphere, or modeling motion via Grassmannian manifolds [10].

We demonstrate our algorithm in three studies of shape. By applying our algo-
rithm to Bookstein’s classical rat skull growth dataset [11], we show that we are
able to obtain a parametric regression curve of similar quality to that produced
by non-parametric methods [12]. We also demonstrate in a 2D corpus callosum
aging study that, in addition to providing more flexibility in the traced path,
our polynomial model provides information about the optimal parametrization
of the time variable. Finally, the usefulness of shape regression for 3D shapes is
demonstrated in an infant cerebellum growth study.

2 Methods

2.1 Preliminaries

Let \((M, g)\) be a Riemannian manifold [13]. For each point \(p \in M\), the metric
\(g\) determines an inner product on the tangent space \(T_pM\) as well as a way to